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JHEP09(2009)034

(http://iopscience.iop.org/1126-6708/2009/09/034)

Published by IOP Publishing for SISSA

Received: *June 5, 2009*
Accepted: *August 19, 2009*
Published: *September 4, 2009*

Weak field black hole formation in asymptotically AdS spacetimes

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Abstract: We use the AdS/CFT correspondence to study the thermalization of a strongly coupled conformal field theory that is forced out of its vacuum by a source that couples to a marginal operator. The source is taken to be of small amplitude and finite duration, but is otherwise an arbitrary function of time. When the field theory lives on $R^{d-1,1}$, the source sets up a translationally invariant wave in the dual gravitational description. This wave propagates radially inwards in AdS_{d+1} space and collapses to form a black brane. Outside its horizon the bulk spacetime for this collapse process may systematically be constructed in an expansion in the amplitude of the source function, and takes the Vaidya form at leading order in the source amplitude. This solution is dual to a remarkably rapid and intriguingly scale dependent thermalization process in the field theory. When the field theory lives on a sphere the resultant wave either slowly scatters into a thermal gas (dual to a glueball type phase in the boundary theory) or rapidly collapses into a black hole (dual to a plasma type phase in the field theory) depending on the time scale and amplitude of the source function. The transition between these two behaviors is sharp and can be tuned to the Choptuik scaling solution in $R^{d,1}$.

Keywords: Gauge-gravity correspondence, Black Holes in String Theory, AdS-CFT Correspondence

ArXiv ePrint: 0904.0464

doi:10.1088/1126-6708/2009/09/034

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1 Introduction

The AdS/CFT correspondence identifies asymptotically AdS gravitational dynamics with the master field evolution of 'large N' field theories. In particular, it relates the evolution of spacetimes with horizons to the non equilibrium statistical dynamics of the high temperature phase of the dual field theory. This connection has recently been studied in detail in a near equilibrium limit. It has been established that the spacetimes that locally (i.e. tube wise) approximate the black brane metric obey the equations of boundary fluid dynamics with gravitationally determined dissipative constants.[1] The equations of fluid dynamics are thus embedded in a long distance sector of asymptotically AdS gravity, a fascinating connection that promises to prove useful in many ways.

Given the success in using gravitational physics to study near equilibrium field theory dynamics, it is natural to attempt to use gravitational dynamics to study far from equilibrium field theory processes. In this paper we will study the gravitational dual of the process of equilibration; i.e the dynamical passage of a system from a pure state in its 'low temperature' phase to an approximately thermalized state in its high temperature phase (see [35–48] for closely related earlier work and [49–52] for analyses of thermalization directly in large N gauge theories). As has been remarked by several authors, this process is dual to the gravitational process of black hole formation via gravitational collapse. The dynamical process is fascinating in its own right, but gains additional interest in asymptotically AdS spaces because of its link to field theory equilibration dynamics. In this paper we study asymptotically AdS (and briefly asymptotically flat) collapse processes in a weak field limit that displays rich dynamics while allowing for analytic control.

[1] See [1–20] for a recent structural understanding of this connection. See the reviews [21–23]for references to important earlier work. See also [24–34] for related line of development.

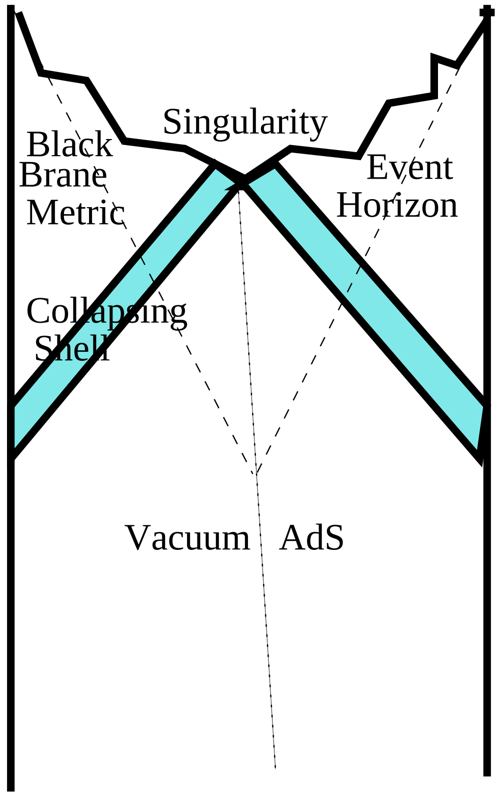

Figure 1. Cross section of the causal diagram for the collapse process in an asymptotically global AdS_{d+1} space. The conventional Penrose diagram for this process would include only the half of the diagram that to the right of its vertical axis of symmetry

An AdS collapse process that could result in black hole formation may be set up, following Yaffe and Chesler [53], as follows . Consider an asymptotically locally AdS spacetime, and let $\mathcal{R}$ denote a finite patch of the conformal boundary of this spacetime. We choose our spacetime to be exactly AdS outside the causal future of $\mathcal{R}$. On $\mathcal{R}$ we turn on the non normalizable part of a massless bulk field. This boundary condition sets up an ingoing shell of the corresponding field that collapses in AdS space. Under appropriate conditions the subsequent dynamics can result in black hole formation.

In this paper we will study the AdS collapse scenario (plus a flat space counterpart) outlined in the previous paragraph in a weak field limit; i.e. we always choose the amplitude ϵ of the non normalizable perturbation to be small. In the interest of simplicity we also focus on situations that preserve a great deal of symmetry, as we explain further below. In the rest of this introduction we describe the three classes of collapse situations we study, and the principal results of our analysis.[2]

[2]In most of the bulk of the text of this paper we only present formulae for asymptotically AdS_{d+1} spacetimes for the smallest nontrivial value of d namely $d = 3$. As we explain in the appendix B however, most of the qualitative results of our analysis apply to arbitrary odd d for $d \geq 3$ and also plausibly to arbitrary even d for $d \geq 4$.

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1.1 Translationally invariant asymptotically AdS_{d+1} collapse

In the first part of this paper we analyze spacetimes that asymptote to Poincare patch AdS_{d+1} space and turn on non normalizable modes on the boundary.[3] We choose our non normalizable data to depend on the boundary time but to be independent of boundary spatial coordinates. Moreover, our data has support only in the time interval $v \in (0, \delta t)$, i.e. our forcing functions are turned on only over a limited time interval. Our boundary conditions create a translationally invariant wave of small amplitude ϵ near the boundary of AdS. This wave then propagates into the bulk of AdS space.

In section 2 and appendices A and B.1 of this paper we demonstrate that this wave *always* results in black brane formation at small amplitude (see figure 1 for the Penrose diagram of the analogous process in an asymptotically global AdS space). Outside the event horizon, this black brane formation process is reliably described by a perturbation expansion in the amplitude. At leading order in perturbation theory the spacetime set up by this wave takes the Vaidya form[4] ([59–61], see e.g. [62] for a review)

$$ds^2 = 2drdv - \left(r^2 - \frac{M(v)}{r^{d-2}}\right) dv^2 + r^2 dx_i^2. \tag{1.1}$$

This form of the metric is exact for all r when $v < 0$, and is a good approximation to the metric for $r \gg \frac{\epsilon^{\frac{2}{d-1}}}{\delta t}$ when $v > 0$. Our perturbative procedure determines the function $M(v)$ in (1.1) in terms of the non normalizable data at the boundary; $M(v)$ turns out to be of order $\frac{\epsilon^2}{(\delta t)^d}$.[5] $M(v)$ reduces to constant M for $v > \delta t$ in odd d and asymptotes to that value (like a power in $\frac{\delta t}{v}$) in even d.[6] In either case the spacetime (1.1) describes the process of formation of a black brane of temperature $T \sim \frac{\epsilon^{\frac{2}{d}}}{\delta t}$ over the time scale of order δt. Note that the time scale of formation of the brane is much smaller than its inverse temperature. This fact allows us to compute the event horizon of the spacetime (1.1) in a simple and explicit fashion in a power series in $\delta t T \sim \epsilon^{\frac{2}{d}}$. To leading order in ϵ the event horizon manifold is given by

$$\begin{aligned} r_H(v) &= M^{\frac{1}{d}} && v > 0 \\ r_H(v) &= \frac{M^{\frac{1}{d}}}{1 - M^{\frac{1}{d}} v} && v < 0 \end{aligned} \tag{1.2}$$

[3] See [54–57] for other work on Poincare patch AdS solutions forced by time dependent non normalizable data

[4] The Vaidya metric is an exact solution for the propagation of a null dust - a fluid whose stress tensor is proportional to $\rho k_\mu k_\nu$ for a lightlike vector k_μ ($k_\mu = \partial_r$ in (1.1)). Note that $\rho k_\mu k_\nu$ is also the stress tensor of a massless field in the eikonol or geometric optics approximation. The AdS-Vaidya metric has been studied before in the context of the AdS/CFT correspondence in, for instance, [58].

[5] More precisely, let $\phi_0(v) = \epsilon\ \chi(\frac{v}{\delta t})$ where χ is a function that is defined on $(0, 1)$. Then the energy of the resultant black brane is $\frac{\epsilon^2}{(\delta t)^d} \times A[\chi]$ where $A[\chi]$ is a functional of $\chi(x)$ that is computed later in this paper.

[6] $M(v)$ is defined as the coefficient of the $\frac{dv^2}{r^{d-2}}$ term in the metric, in an expansion around small r. In even d this turns out not to be equal to the mass density of the system, the coefficient of the same term in the metric when expanded around large r.

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All of the spacetime outside the event horizon (1.2) lies within the domain of validity of our perturbative procedure. Of course perturbation theory does not accurately describe the process of singularity formation of the black brane. However the region where perturbation theory breaks down (and so (1.1) is not reliable) is contained entirely within the event horizon of (1.1). Consequently, the region outside perturbative control is causally disconnected from physics outside the event horizon, so our perturbation procedure gives a fully reliable description of the dynamics outside the event horizon. It follows in particular that any singularities that develop in our solution are is always shielded by a regular event horizon, in agreement with the cosmic censorship conjecture.[7]

In section 2 we demonstrate that the corrections to the Vaidya metric (1.1) may be systematically computed in a power series in positive fractional powers of ϵ. At any order in the perturbation expansion, the metric may be determined analytically for times $v \ll T^{-1}$ (T is the temperature of the eventually formed brane). At times of order or larger than T^{-1}, perturbative corrections to the metric are determined in terms solutions of universal(i.e. independent of the form of the perturbation) linear differential equations which we have only been able to solve numerically. Even at late times, however our perturbative procedure analytically determines the dependence of observables on the functional form of the non normalizable perturbation, allowing us to draw conclusions that are valid for small amplitude perturbation of arbitrary form.

Let us now word our results in dual field theoretic terms. Our gravity solution describes a CFT initially in its vacuum state. Over the time period $(0, \delta t)$ the field theory is perturbed by a translationally invariant time dependent source, of amplitude ϵ, that couples to a marginal operator. This coupling pumps energy into this system. Our perturbative gravitational solution gives a detailed description of the subsequent equilibration process; in particular it gives a precise formula for the temperature of the final equilibrium configuration as a function of the perturbation function. It also, very surprisingly, asserts that for some purposes[8] our system appears to thermalize almost instanteneously at leading order in ϵ. We pause to explain this in detail.[9]

A field theorist presented with a flow towards equilibrium might choose to probe this flow by perturbing it with an infinitesimal source, localized at some time. He would then measure the subsequent change in the solution in response to this perturbation. However note that the spacetime in (1.1) is identical to the spacetime outside a static uniform black brane for $v > \delta t$ when d is odd (and for $v \gg \delta t$ for even d). It follows that the response of our system to any boundary perturbation localized at times $v > \delta t$ in odd d (and at $v \gg \delta t$ in even d) will be identical to the response of a thermally equilibrated system to the same perturbation. In other words our system responds to perturbations at $v > \delta t$ as if it had equilibrated instanteneously.

A field theorist could also characterize a flow towards equilibrium by recording the

[7] We thank M. Rangamani for discussions on this point.

[8] In particular, in even bulk space time dimensions, one point functions of all local operators reduce to their thermal values as soon as the perturbation is switched off. While thermalization of one point functions is not instantaneous in odd bulk space times it appears to take place over a time scale of order δt.

[9] The rest of this subsection was worked out in collaboration with O. Aharony, B. Kol and S. Raju. See also the paper [48], by Lin and Shuryak, for a very similar earlier discussion. We thank E. Shuryak for bringing this paper to our attention.

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values of all observables as a function of time (in the absence of any further perturbation). The full set of observables consists of expectation values of the arbitrary product of 'gauge invariant' operators, i.e. quantities that in a gauge theory would take the form

$$\langle TrO_1 TrO_2 \dots TrO_n \rangle.$$

In this paper we work in the strict large N limit (i.e. the strictly classical limit from the dual bulk viewpoint). In this limit trace factorization (or the classical nature of the dual bulk theory) ensures that the expectation value of products equals the product of expectation values. In other words our set of observables is given precisely by the one point functions of all gauge invariant operators.

Now note that expectation values of all *local* boundary operators are determined by the bulk solution in a neighborhood of the boundary values. As the metric (1.1) is identical to the metric of a uniform black brane in the neighbourhood of the boundary when $v > \delta t$, it follows that the expectation value of all *local* boundary operators reduce instantaneously to their thermal values in odd d (and when $v \gg \delta t$ in even d). Consequently, all local operators appear to thermalize instanteneously.

Not all gauge invariant operators are local, however. A field theorist could also record the values of non local observables, like circular Wilson Loops of radius a, as a function of time. As nonlocal observables probe the spacetime away from the boundary, their expectation values reduce to thermal results only after a larger time that depends on the size of the loop (this time is proportional to a at small a). So a diligent infinite N field theorist would be able to distinguish (1.1) from absolute thermal equilibrium at times greater than δt, but only by keeping track of the expectation values of non local observables.

If one were to retreat away from the large N limit one would find large new classes of gauge invariant observables; the connected correlators of, for instance, local gauge invariant operators. Such correlators also sample spacetime away from the boundary, the distance scale of this nonlocal sampling being set by the separation between the operator insertions (see [58] for a detailed discussion of properties of correlation functions in asymptotically AdS Vaidya type metrics). As in our discussion of Wilson loops above, the time scale for thermalization of such connected correlators is set by their separation (it is proportional to their separation when this separation is small).

As we have seen, the time scale of equlibration of the solutions described in this paper depend on the precise question you ask about it. We would now like to describe a concrete and possibly practically important experimental sense in which our system behaves as if it were instanteneously thermally equilibrated.

Consider the response of a CFT in its vacuum to a forcing function that varies — though only slowly — with $\vec{x}$. We anticipate that at $v = \delta t$ the corresponding spacetime is locally (tube wise) well described by a black brane metric with a value of the temprature that varies with $\vec{x}$ (see (5.1) and the discusson arounf it in section 5). According to [1–20], the subsequent evolution of our system is governed by the equations of boundary fluid dynamics. The initial conditions for the relevant fluid flow are given at $v = \delta t$. Consequently an experimentalist who observes the subsequent fluid flow, and back calculates, would conclude that his system was thermalized at $v = \delta t$.

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The thought experiment of the previous paragraph is reminiscent of situation at the RHIC experiment. The back calculation described in this paragraph, in the context of that experiment, suggests that the RHIC system is governed by fluid dynamics at times of order 0.5 fermi after the collision, much faster than suggested by naive estimates for thermalization time (see [63] and references therein). It is natural to wonder whether the mechanisims for rapid equilibration of this paper have qualitative applicability to the RHIC experiment. We leave a serious investigation of this question to future work.

In summary, (1.1) describes a system whose response to additional external perturbations at $v > \delta t$ is identical to that of a thermally equilibrated system and whose one point functions of local operators also instanteneously thermalize. However expectation values of non local observables (or correlators) thermalize more slowly, over a time scale that depends on the smearing size of the observable (or correlator). We find instantenous thermalization of expectation values local operators and the scale dependence in the process of equilibration fascinating. In fact this discussion is reminiscent of precursors in the AdS/CFT correspondence [64–67].

We emphasize that our discussion of thermalization applies only at leading order in ϵ expansion. Indeed our analysis was based on (1.1) which accurately describes our spacetime only at leading order in ϵ. At sub leading orders (1.1) is corrected by perturbations that decay to the black brane result only over the time scale $1/T$, in accordance with naive expextation. Consequently, the instantaneous thermalization of expectation values of local operators is corrected by sub leading equilibration process that take place over the time scale $1/T$, the thermalization of linear fluctuations about a brane of temperature T. Note, in particular, that we have no reason so suspect that thermalization occurs over a time period that is faster than the naive estimate $v = \frac{1}{T}$ when ϵ is of order unity or larger.

1.2 Spherically symmetric collapse in flat space

We next turn to the perturbative study of spherically symmetric collapse in an asymptotically flat space. Consider a spherically symmetric shell, propagating inwards, focused onto the origin of an asymptotically flat space. Such a shell may qualitatively be characterized by its thickness and mass, or (more usefully for our purposes) by the Schwarzschild radius r_H associated with this mass. It is a well appreciated fact that this collapse process may reliably be described in an amplitude expansion when $y \equiv \frac{r_H}{\delta t}$ is very small. The starting point for this expansion is the propagation of a free scalar shell. This free motion receives weak scattering corrections at small y, which may be computed perturbatively.

In section 3 of this paper we demonstrate that this flat space collapse process may also be reliably described in an amplitude expansion at *large* y. In section 3 and appendix B.2 we study this collapse process mainly in odd d (i.e. in even bulk spacetime dimensions). The starting point for this expansion is a Vaidya metric similar to (1.1), whose event horizon we are able to reliably compute in a power series expansion in inverse powers of y. Outside this event horizon the dilaton is everywhere small and the Vaidya metric receives only weak scattering corrections that it may systematically be computed in a power series in $\frac{1}{y}$ at large y. As in the previous subsection, our perturbative procedure is not valid everywhere;

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however the breakdown of perturbation theory occurs entirely within the event horizon, and so does not impinge on our control of the solution outside the event horizon.

At early times we are able to determine the perturbative corrections to the metric (order by order in $\frac{1}{y}$) in an entirely analytic manner. However late time corrections to the metric are computed in terms of the solutions to relevant universal linear differential equations, which we have not been able to solve analytically. However our perturbative solutions carry a considerable amount of information, even in the absence of an explicit analytic solution to the relevant differential equation. As an example, in section 3 we determine the fraction of energy of the incident pulse that is radiated back out to infinity to nontrivial leading order in the expansion in $\frac{1}{y}$. We are able to analytically determine the dependence of this fraction on the shape of the incident pulse upto an overall constant (see (3.34)). The determination of the value of this constant requires knowledge of the explicit solution of the 'universal' differential equation listed in section 3, and may presumably be determined numerically.

An order parameter (the presence of an event horizon at late times) distinguishes small y from large y behavior, so the transition between them must be sharp. This observation was originally made about twenty years ago in classic paper by Christodoulou (see [68]) and references therein) who rigorously demonstrated that collapse at arbitrarily large y results in black hole formation, while collapse at small y does not. As the fascinating transition between small and large y behaviors (which has been extensively in a programme of numerical relativity initiated by Choptuik [69])[10] presumably occurs at y of order unity. Consequently it cannot be studied in either the small y or the large y expansions described in our paper.

As we do not have a holographic description of gravitational dynamics in an asymptotically flat space, we are unable to give a direct dual field theoretic interpretation of our results reviewed in this subsection. See however, the next subsection.

1.3 Spherically symmetric collapse in asymptotically global AdS

The process of spherically symmetric collapse in an asymptotically global AdS space constitutes an interesting one parameter interpolation between the collapse processes described in subsections 1.1 and 1.2. We study such collapse processes in section 4 of this paper. In section 4 we have studied this collapse situation in detail only in $d = 3$. In this subsection we report the generalization of these results to arbitrary odd dimension, which may qualitatively be inferred from the results of appendix B.

Consider a global AdS space, whose boundary is taken to be a sphere of radius R $\times$ time. Consider a collapse process initiated by radially symmetric non normalizable boundary conditions that are turned on, uniformly over the boundary sphere, over a time interval δt. The amplitude ϵ of this perturbation together with the dimensionless ratio $x \equiv \frac{\delta t}{R}$, constitute the two qualitatively important parameters of this perturbation. In the limit $x \to 0$ it is obvious that the collapse process of this subsection effectively reduces to

[10] See [70, 71] for reviews and [72–77] for recent work interpreting this transition in the context of the AdS/CFT correspondence.

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the Poincare patch collapse process described in subsection 1.1, and results in the formation of a black hole that is large compared to the AdS radius (and so locally well approximates a flat black brane); quantitatively this turns out to work for $x \ll \epsilon^{\frac{2}{d}}$. When $x \gg \epsilon^{\frac{2}{d}}$ the most interesting part of the collapse process takes place in a bubble of approximately flat space. In this case the solution closely resembles a wave propagating in AdS space at large r, glued onto a flat space collapse process described in subsection 1.2.[11] Following through the details of the gluing process, it turns out that the inverse of the effective flat space y parameter (see subsection 1.2) is given by $\frac{x^{\frac{2d-2}{d-2}}}{\epsilon^{\frac{2}{d-2}}}$. The parameter y is of order unity when $x \sim \epsilon^{\frac{1}{d-1}}$. We conclude that the end point of the global AdS collapse process is a black hole for $x \ll \epsilon^{\frac{1}{d-1}}$ but a scattering dilaton wave for $x \gg \epsilon^{\frac{1}{d-1}}$.

The minimum mass of black holes formed through this process is $\frac{\epsilon^{\frac{d-2}{d-1}}}{R}$ (we work in units in which the mass of the black hole is simply the long time value of the parameter M in (4.10), the global analogue of (1.1)). Let us contrast this with the minimum mass of black holes that we expect to be produced when we pump energy into the more slowly (i.e. through a forcing function whose time variation is of order $\frac{1}{R}$) but over a long time period. As we have described above, slow forcing deposits energy into the gravitational thermal gas. By continually forcing the system one creates a thermal gas of increasing energy. At a critical energy density of order $\frac{1}{R}$, however, density fluctuations in this thermal gas become unstable [78]; the end point of this instability is believed to be a black hole. Clearly this slow pumping in of energy produces black holes of energy $\frac{1}{R}$ or greater. It follows that black hole production can be produced more efficiently (i.e. at lower energies) via rapid forcing than via a slow pumping in of energy into the system.

As we have explained above, when $\epsilon \ll 1$ and when $x \ll \epsilon^{\frac{1}{d-1}}$, we are able to reliably establish black hole formation within perturbation theory (see figure 1 for a Penrose diagram of this process). As in the previous two subsections, the starting point of the perturbative expansion always turns out to be a metric of the Vaidya form, whose event horizon we are able to reliably compute. Our metric receives only small scattering corrections outside the event horizon. Although the perturbative procedure breaks down badly near the black hole singularity, that is irrelevant for the physics outside the event horizon.

On the other hand, when $x \gg \epsilon^{\frac{1}{d-1}}$ (but at small ϵ), the incident waves simply scatter through the origin, and subsequently undergo periodic motion in AdS space. This free motion is corrected by interaction effects that will eventually cause this dilaton pulse to

[11]This statement is only correct at times $v \ll \frac{1}{R}$. To see why recall that when a collapsing shell in flat space forms a black hole, some of its energy is radiated out to $\mathcal{I}^+$. The resolution of the infalling shell into a static black hole and plus a shell radiated out to infinity occurs over a time scale set by r_H the Schwarszshild radius associated with the infalling matter. In AdS space this shell eventually reflects off the boundary of AdS space at times of order $\frac{1}{R}$ (note that this is a much larger length scale than r_H when the black hole is small enough) and then is refocussed on the origin of space. This process repeats itself unendingly; eventaully all of the energy of the intitial shell is absorbed by the black hole. Consequently AdS collapse processes always differ significantly from their flat space counterparts for $v \ll \frac{1}{R}$. In particular, while such a process can result in the formation of arbitrarily small mass black holes over time scale $\frac{1}{R}$, the mass of black holes created at long times is bounded from below (see below for an estimate). We thank V. Hubeny for a discussion on this point.

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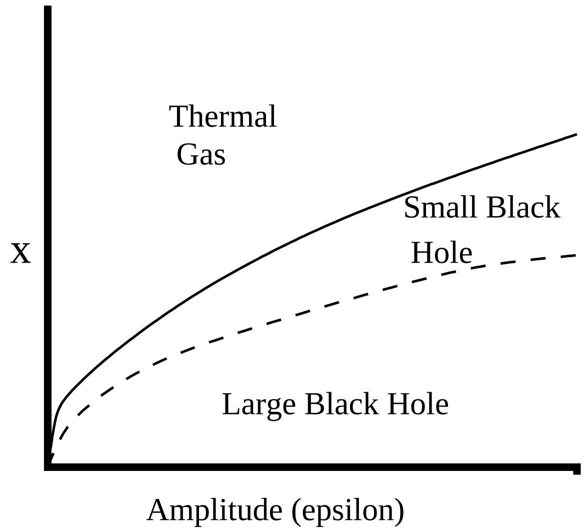

Figure 2. The 'Phase Diagram' for our dynamical stirring in global AdS. The final outcome is a large black hole for $x \ll \epsilon^{\frac{2}{d}}$ (below the dashed curve), a small black hole for $x \ll \epsilon^{\frac{1}{d-1}}$ (between the solid and dashed curve) and a thermal gas for $x \gg \epsilon^{\frac{1}{d-1}}$. The solid curve represents non analytic behaviour (a phase transition) while the dashed curve is a crossover.

deviate significantly from its free motion over a time scale that we expect to scale like a positive power of $\frac{x^{\frac{2d-2}{d-2}}}{\epsilon^{\frac{2}{d-2}}}$ times the inverse radius of the sphere.[12]

Let us now reword our results in field theory terms. Any CFT that admits a two derivative gravity dual description undergoes a first order finite temperature phase transition when studied on S^{d-1}. The low temperature phase is a gas of 'glueballs' (dual to gravitons) while the high temperature phase is a strongly interacting, dissipative, 'plasma' (dual to the black hole). The gravitational solutions of this paper describe such a CFT on a sphere, initially in its vacuum state. We then excite the CFT over a time δt by turning on a spherically symmetric source function that couples to a marginal operator. The most important qualitative question about the subsequent equilibration process is: in which phase does the system eventually settle down within classical dynamics (i.e. ignoring tunneling effects) ? Our gravitational solutions predict that the system settles in its free particle phase when $x \gg \epsilon^{\frac{1}{d-1}}$ but in the plasma phase when $x \ll \epsilon^{\frac{1}{d-1}}$. As in subsection 1.1 the equilibration in the high temperature phase is almost instantaneous. However equilibration in the low temperature phase appears to occur over a much longer time scale. We note also that the transition between these two end points appears to be singular (this is the Choptuik singularity) in the large N limit.[13] This singularity is presumably smoothed out by fluctuations at finite N, a phenomenon that should be dual to the smoothing out of a naked gravitational singularity by quantum gravity fluctuations.

[12] We expect this pulse to thermalize over an even longer time scale, one that scales as a positive power of the larger number, $\frac{x^d}{\epsilon^2}$.

[13] See section 5 for a discussion of the effects of potential Gregory-Laflamme type instabilities near this singular surface.

In the rest of this paper and in the appendices we will present a detailed study of the collapse scenarios outlined in this introduction. In the last section of this paper we also present a discussion of our results.

2 Translationally invariant collapse in AdS

In this section we study asymptotically planar (Poincare patch) AdS_{d+1} solutions to negative cosmological constant Einstein gravity interacting with a minimally coupled massless scalar field (note that this system obeys the null energy condition). We focus on solutions in which the boundary value of the scalar field takes a given functional form $\phi_0(v)$ in the interval $(0, \delta t)$ but vanishes otherwise. The amplitude of $\phi_0(v)$ (which we schematically refer to as ϵ below) will be taken to be small in most of this section. The boundary dual to our setup is a d dimensional conformal field theory on $R^{d-1,1}$, perturbed by a spatially homogeneous and isotropic source function, $\phi_0(v)$, multiplying a marginal scalar operator.

Note that our boundary conditions preserve an $R^{d-1} \times \mathrm{SO}(d-1)$ symmetry (the R^{d-1} factor is boundary spatial translations while the $\mathrm{SO}(d-1)$ is boundary spatial rotations). In this section we study solutions on which $R^{d-1} \rtimes \mathrm{SO}(d-1)$ lifts to an isometry of the full bulk spacetime. In other words the spacetimes studied in this section preserve the maximal symmetry allowed by our boundary conditions. As a consequence all bulk fields in our problem are functions of only two variables; a radial coordinate r and an Eddington Finkelstein ingoing time coordinate v. The chief results of this section are as follows:

- The boundary conditions described above result in black brane formation for an arbitrary (small amplitude) source functions $\phi_0(v)$.
- Outside the event horizon of our spacetime, we find an explicit analytic form for the metric as a function of $\phi_0(v)$. Our metric is accurate at leading order in the ϵ expansion, and takes the Vaidya form (1.1) with a mass function that we determine explicitly as a function of time.
- In particular, we find that the energy density of the resultant black brane is given, to leading order, by

  $$C_2 = \frac{2^{d-1}}{(d-1)} \left(\frac{(\frac{d-1}{2})!}{(d-1)!} \right)^2 \int_{-\infty}^{\infty} \left(\left(\partial_t^{\frac{d+1}{2}} \phi_0(t) \right)^2 \right) \tag{2.1}$$

  in odd d and by

  $$C_2 = -\frac{d^2}{(d-1)2^d} \frac{1}{\left(\frac{d}{2}!\right)^2} \int dt_1 dt_2 \partial_{t_1}^{\frac{d+2}{2}} \phi_0(t_1) \ln(t_1 - t_2) \theta(t_1 - t_2) \partial_{t_2}^{\frac{d+2}{2}} \phi_0(t_2) \tag{2.2}$$

  in even d. Note that, in each case, $C_2 \sim \frac{\epsilon^2}{(\delta t)^d}$.
- We find an explicit expression for the event horizon of the resultant solutions, at leading order, and thereby demonstrate that singularities formed in the process of black brane formation are always shielded by a regular event horizon at small ϵ.

- Perturbation theory in the amplitude ϵ yields systematic corrections to this leading order metric. We unravel the structure of this perturbation expansion in detail and compute the first corrections to the leading order result.

While every two derivative theory of gravity that admits an AdS solutions admits a consistent truncation to Einstein gravity with a negative cosmological constant, the same statement is clearly not true of gravity coupled to a minimally coupled massless scalar field. It is consequently of considerable interest to note that results closely analogous to those described above also apply to the study of Einstein gravity with a negative cosmological constant. In appendix A we analyze the process of black brane formation by gravitational wave collapse in the theory of pure gravity (similar to the set up of [53]), and find results that are qualitatively very similar to those reported in this section. The solutions of appendix A yield the dual description of a class of thermalization processes in every 3 dimensional conformal field theory that admits a dual description as a two derivative theory of gravity. In fact, the close similarity of the results of appendix A with those of this section, lead us to believe that the results reported in this section are qualitatively robust. In particular we think it is very likely that results of this section will qualitatively apply to the most general small amplitude translationally invariant collapse process in the systems we study.

2.1 The set up

Consider a minimally coupled massless scalar (the 'dilaton') interacting with negative cosmological constant Einstein gravity in $d+1$ spacetime dimensions

$$S = \int d^{d+1}x\sqrt{g}\left(R - \frac{d(d-1)}{2} - \frac{1}{2}(\partial\phi)^2\right) \tag{2.3}$$

The equations of motion that follow from the Lagrangian (2.3) are

$$\begin{aligned} &E_{\mu\nu} \equiv G_{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left(-\frac{d(d-1)}{2} + \frac{1}{4}(\partial\phi)^2\right) = 0 \\ &\nabla^2\phi = 0 \end{aligned} \tag{2.4}$$

where the indices μ, ν range over all $d+1$ spacetime coordinates. As mentioned above, in this section we are interested in locally asymptotically AdS_{d+1} solutions to these equations that preserve an $R^{d-1} \times \mathrm{SO}(d-1)$ symmetry group. This symmetry requirement forces the boundary metric to be Weyl flat (i.e. Weyl equivalent to flat $R^{d-1,1}$); however it allows the boundary value of the scalar field to be an arbitrary function of boundary time v. We choose this function as

$$\begin{aligned} &\phi_0(v) = 0 \quad (v < 0) \\ &\phi_0(v) < \epsilon \quad (0 < v < \delta t) \\ &\phi_0(v) = 0 \quad (v > \delta t) \end{aligned} \tag{2.5}$$

(we also require that $\phi_0(v)$ and its first few derivatives are everywhere continuous.[14]).

[14] We expect that all our main physical conclusions will continue to apply if we replace our ϕ_0 — which is chosen to strictly vanish outside $(0, \delta t)$ — by any function that decays sufficiently rapidly outside this range.

Everywhere in this paper we adopt the 'Eddington Finkelstein' gauge $g_{rr} = g_{ri} = 0$ and $g_{rv} = 1$. In this gauge, and subject to our symmetry requirement, our spacetime takes the form

$$\begin{aligned} ds^2 &= 2drdv - g(r,v)dv^2 + f^2(r,v)dx_i^2 \\ \phi &= \phi(r,v). \end{aligned} \tag{2.6}$$

The mathematical problem we address in this subsection is to solve the equations of motion (2.4) for the functions ϕ, f and g, subject to the pure AdS initial conditions

$$\begin{aligned} g(r,v) &= r^2 \quad (v<0) \\ f(r,v) &= r \quad\;\; (v<0) \\ \phi(r,v) &= 0 \quad\;\; (v<0) \end{aligned} \tag{2.7}$$

and the large r boundary conditions

$$\begin{aligned} &\lim_{r\to\infty} \frac{g(r,v)}{r^2} = 1 \\ &\lim_{r\to\infty} \frac{f(r,v)}{r} = 1 \\ &\lim_{r\to\infty} \phi(r,v) = \phi_0(v) \end{aligned} \tag{2.8}$$

The Eddington Finkelstein gauge we adopt in this paper does not completely fix gauge redundancy (see [53] for a related observation). The coordinate redefinition $r = \tilde{r} + h(v)$ respects both our gauge choice as well as our boundary conditions. In order to completely define the mathematical problem of this section, we must fix this ambiguity. We have assumed above that $f(r,v) = r + \mathcal{O}(1)$ at large r. It follows that under the unfixed diffeomorphism, $f(r,v) \to f(r,v) + h(v) + \mathcal{O}(1/r)$. Consequently we can fix this gauge redundancy by demanding that $f(r,v) \approx r + \mathcal{O}(1/r)$ at large r. We make this choice in what follows. As we will see below, it then follows from the equations of motion that $g(r) = r^2 + \mathcal{O}(1)$. Consequently, the boundary conditions (2.8) on the fields g, f and ϕ, may be restated in more detail as

$$\begin{aligned} g(r,v) &= r^2\left(1 + \mathcal{O}\left(\frac{1}{r^2}\right)\right) \\ f(r,v) &= r\left(1 + \mathcal{O}\left(\frac{1}{r^2}\right)\right) \\ \phi(r,v) &= \phi_0(v) + \mathcal{O}\left(\frac{1}{r}\right) \end{aligned} \tag{2.9}$$

Equations (2.4), (2.6), (2.7) and (2.9) together constitute a completely well defined dynamical system. Given a particular forcing function $\phi_0(v)$, these equations and boundary conditions uniquely determine the functions $\phi(r,v)$, $g(r,v)$ and $f(r,v)$.

2.2 Structure of the equations of motion

The nonzero equations of motion (2.4) consist of four nontrivial Einstein equations E_{rr}, E_{rv}, E_{vv} and $\sum_i E_{ii}$ (where the index i runs over the $d-1$ spatial directions) together

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with the dilaton equation of motion. For the considerations that follow below, we will find it convenient to study the following linear combinations of equations

$$\begin{aligned} E_c^1 &= g^{v\mu} E_{\mu r} \\ E_c^2 &= g^{v\mu} E_{\mu v} \\ E_{\rm ec} &= g^{r\mu} E_{\mu r} \\ E_{\rm d} &= \sum_{i=1}^{d-1} E_{ii} \\ E_\phi &= \nabla^2 \phi \end{aligned} \tag{2.10}$$

Note that the equations E_c^1 and E_c^2 are constraint equations from the point of view of v evolution.

It is possible to show that E_d and $\frac{d(rE_{\rm ec})}{dr}$ both automatically vanish whenever $E_c^1 = E_c^2 = E_\phi = 0$. This implies that this last set of three independent equations — supplemented by the condition that $rE_{\rm ec} = 0$ at any one value of r — completely exhaust the dynamical content of (2.4). As a consequence, in the rest of this paper we will only bother to solve the two constraint equations and the dilaton equation, but take care to simultaneously ensure that $rE_{\rm ec} = 0$ at some value of r. It will often prove useful to impose the last equation at arbitrarily large r. This choice makes the physical content of $rE_{\rm ec} = 0$ manifest; this is simply the equation of energy conservation in our system.[15]

2.2.1 Explicit form of the constraints and the dilaton equation

With our choice of gauge and notation the dilaton equation takes the minimally coupled form

$$\partial_r \left(f^{d-1} g \partial_r \phi \right) + \partial_v \left(f^{d-1} \partial_r \phi \right) + \partial_r \left(f^{d-1} \partial_v \phi \right) = 0 \tag{2.11}$$

Appropriate linear combinations of the two constraint equations take the form

$$\begin{aligned} (\partial_r \phi)^2 &= -\frac{2(d-1)\partial_r^2 f}{f} \\ \partial_r \left(f^{d-2} g \partial_r f + 2 f^{d-2} \partial_v f \right) &= f^{d-1} d \end{aligned} \tag{2.12}$$

Note that the equations (2.12) (together with boundary conditions and the energy conservation equation) permit the unique determination of $f(r, v_0)$ and $g(r, v_0)$ in terms of $\phi(r, v_0)$ and $\dot{\phi}(r, v_0)$ (where v_0 is any particular time). It follows that f and g are not independent fields. A solution to the differential equation set (2.11) and (2.12) is completely specified by the value of ϕ on a constant v slice (note that the equations are all first order in time derivatives, so $\dot{\phi}$ on the slice is not part of the data of the problem) together with the boundary condition $\phi_0(v)$.

[15]It turns out that both E_d and the dilaton equation of motion are automatically satisfied whenever $E_{\rm ec}$ together with the two Einstein constraint equations are satisfied. Consequently $E_{\rm ec}$ plus the two Einstein constraint equations form another set of independent equations. This choice of equations has the advantage that it does not require the addition of any additional condition analogous to energy conservation. However it turns out to be an inconvenient choice for implementing the ϵ expansion of this paper, and we do not adopt it in this paper.

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2.3 Explicit form of the energy conservation equation

In this section we give an explicit form for the equation $E_{\rm ec} = 0$ at large r. We specialize here to $d = 3$ but see appendix B.1 for arbitrary d. Using the Graham Fefferman expansion to solve the equations of motion in a power series in $\frac{1}{r}$ we find

$$\begin{aligned} f(r,v) &= r\left(1 - \frac{\dot{\phi_0}^2}{8r^2} + \frac{1}{r^4}\left(\frac{1}{384}(\dot{\phi}_0)^4 - \frac{1}{8}L(v)\dot{\phi}_0\right) + \mathcal{O}\left(\frac{1}{r^5}\right)\right) \\ g(r,v) &= r^2\left(1 - \frac{3(\dot{\phi}_0)^2}{4r^2} - \frac{M(v)}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)\right) \\ \phi(r,v) &= \phi_0(v) + \frac{\dot{\phi}_0}{r} + \frac{L(v)}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) \end{aligned} \tag{2.13}$$

where the functions $M(v)$ and $L(v)$ are undetermined functions of time that are, however, constrained by the energy conservation equation $E_{\rm ec}$, which takes the explicit form

$$\dot{M} = \dot{\phi}_0\left(\frac{3}{8}(\dot{\phi}_0)^3 - \frac{3L}{2} - \frac{1}{2}\dddot{\phi}_0.\right) \tag{2.14}$$

In all the equations in this subsection and in the rest of the paper, the symbol $\dot{P}$ denotes the derivative of P with respect to our time coordinate v. Solving for $M(v)$ we have[16]

$$M(v) = \frac{1}{2}\int_0^v dt\left(\left(\ddot{\phi}_0\right)^2 + \frac{3}{4}\left(\dot{\phi}_0\right)^4 - 3\dot{\phi}_0 L(t)\right) \tag{2.19}$$

2.4 The metric and event horizon at leading order

Later in this section we will solve the equations of motion (2.11), (2.12) and (2.14) in an expansion in powers of ϵ, the amplitude of the forcing function $\phi_0(v)$. In this subsection we simply state our result for the spacetime metric at leading order in ϵ. We then proceed

[16] We note parenthetically that (2.14) may be rewritten as

$$\dot{T}_0^0 = \frac{1}{2}\dot{\phi}_0\mathcal{L} \tag{2.15}$$

where the value $\mathcal{L}$ of the operator dual to the scalar field ϕ and the stress tensor $T_{\alpha\beta}$ are given by

$$\begin{aligned} \mathcal{L} &\equiv \lim_{r\to\infty} r^3\left(\partial_n\phi + \partial^2\phi\right) \\ T^\mu_\nu &= \lim_{r\to\infty} r^3\left(K^\mu_\nu - (K-2)\delta^\mu_\nu - \mathcal{G}^\mu_\nu + \frac{\partial^\mu\phi\partial_\nu\phi}{2} - \frac{(\partial\phi)^2\delta^\mu_\nu}{4}\right). \end{aligned} \tag{2.16}$$

Where

$$\begin{aligned} K^\mu_\nu &= \text{Extrinsic curvature of the constant } r \text{ surfaces}, \quad K = K^\mu_\mu \\ \mathcal{G}^\mu_\nu &= \text{Einstein tensor evaluated on the induced metric of the constant } r \text{ surfaces} \end{aligned} \tag{2.17}$$

yielding

$$\begin{aligned} T^0_0 &= -2T^x_x = -2T^y_y = M(v) \\ \mathcal{L} &= \frac{3}{4}\dot{\phi}_0^3 - 3L(v) - \partial_v^3\phi_0 \end{aligned} \tag{2.18}$$

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to compute the event horizon of our spacetime to leading order in ϵ. We present the computation of the event horizon of our spacetime before actually justifying the computation of the spacetime itself for the following reason. In the subsections below we will aim to construct the spacetime that describes black hole formation only outside the event horizon. For this reason we will find it useful below to have a prior understanding of the location of the event horizon in the spacetimes that emerge out of perturbation theory.

We will show below that to leading order in ϵ, our spacetime metric takes the Vaidya form (1.1). The mass function $M(v)$ that enters this Vaidya metric is also determined very simply. As we will show below, it turns out that $L(v) \sim \mathcal{O}(\epsilon^3)$ on our perturbative solution. It follows immediately from (2.19) that the mass function $M(v)$ that enters the Vaidya metric, is given to leading order by

$$\begin{aligned} M(v) &= C_2(v) + \mathcal{O}(\epsilon^4) \\ C_2(v) &= -\frac{1}{2}\int_{-\infty}^{v} dt \dot{\phi}_0(t) \dddot{\phi}_0(t) \end{aligned} \tag{2.20}$$

(Here C_2 is the approximation to the mass density, valid to second order in the amplitude expansion, see below).

Note that, for $v > \delta t$, $C_2(v)$ reduces to a constant $M = C_2$ given by

$$C_2 = \frac{1}{2}\int_{-\infty}^{\infty} dt \left(\ddot{\phi}_0(t)\right)^2 \sim \frac{\epsilon^2}{(\delta t)^3} \tag{2.21}$$

In the rest this subsection we proceed to compute the event horizon of the leading order spacetime (1.1) in an expansion in $\epsilon^{\frac{2}{3}}$ expansion. Let the event horizon manifold of our spacetime be given by the surface $S \equiv r - r_H(v) = 0$. As the event horizon is a null manifold, it follows that $\partial_\mu S \partial_\nu S g^{\mu\nu} = 0$, and we find

$$\frac{dr_H(v)}{dv} = \frac{r_H^2(v)}{2}\left(1 - \frac{M(v)}{r_H^3(v)}\right) \tag{2.22}$$

As $M(v)$ reduces to the constant $M = C_2$ for $v > \delta t$, it follows that the event horizon must reduce to the surface $r_H = M^{\frac{1}{3}}$ at late times. It is then easy to solve (2.22) for $v < 0$ and $v > \delta t$; we find

$$r_H(v) = M^{\frac{1}{3}}, \qquad v \geq \delta t \tag{2.23}$$

$$r_H(v) = M^{\frac{1}{3}} x\left(\frac{v}{\delta t}\right), \qquad 0 < v < \delta t \tag{2.24}$$

$$\frac{1}{r_H(v)} = -v + \frac{1}{M^{\frac{1}{3}} x(0)}, \qquad v \leq 0 \tag{2.25}$$

$$\tag{2.26}$$

where $x(y)$ obeys the differential equation

$$\begin{aligned} \frac{dx}{dy} &= \alpha \frac{x^2}{2}\left(1 - \frac{M(y\delta t)}{M x^3}\right) \\ \alpha &= M^{\frac{1}{3}} \delta t \sim \epsilon^{\frac{2}{3}} \end{aligned} \tag{2.27}$$

and must be solved subject to the final state conditions $x = 1$ for $y = 1$. (2.27) is easily solved in a perturbation series in α. We set

$$x(y) = 1 + \sum_n \alpha^n x_n(y) \tag{2.28}$$

and solve recursively for $x_n(t)$. To second order we find[17]

$$\begin{aligned} x_1(y) &= -\int_y^1 dz \left(\frac{1 - \frac{M(z\delta t)}{M}}{2} \right) \\ x_2(y) &= -\int_y^1 dz \; x_1(z) \left(1 + \frac{M(z\delta t)}{2M} \right) \end{aligned} \tag{2.29}$$

In terms of which

$$r_H(v) = M^{\frac{1}{d}} \left(1 + \alpha \; x_1\left(\frac{v}{\delta t}\right) + \alpha^2 x_2\left(\frac{v}{\delta t}\right) + \mathcal{O}(\alpha^3) \right) \quad (0 < v < \delta t) \tag{2.30}$$

Note in particular that, to leading order, $r_H(v)$ is simply given by the constant $M^{\frac{1}{3}}$ for all $v > 0$.

2.5 Formal structure of the expansion in amplitudes

In this subsection we will solve the equations (2.11), (2.12) and (2.14) in a perturbative expansion in the amplitude of the source function $\phi_0(v)$. In order to achieve this we formally replace $\phi_o(v)$ with $\epsilon\phi_0(v)$ and solve all equations in a power series expansion in ϵ. At the end of this procedure we can set the formal parameter ϵ to unity. In other words ϵ is a formal parameter that keeps track of the homogeneity of ϕ_0. Our perturbative expansion is really justified by the fact that the amplitude of ϕ_0 is small.

In order to proceed with our perturbative procedure, we set

$$\begin{aligned} f(r,v) &= \sum_{n=0}^{\infty} \epsilon^n f_n(r,v) \\ g(r,v) &= \sum_{n=0}^{\infty} \epsilon^n g_n(r,v) \\ \phi(r,v) &= \sum_{n=0}^{\infty} \epsilon^n \phi_n(r,v) \end{aligned} \tag{2.31}$$

with

$$f_0(r,v) = r, \qquad g_0(r,v) = r^2, \qquad \phi_0(r,v) = 0. \tag{2.32}$$

[17] In this section we only construct the event horizon for the Vaidya metric. The actual metrics of interest to this paper receive corrections away from the Vaidya form, in powers of $M\delta t$. Consequently, the event horizons for the actual metrics determined in this paper will agree with those of this subsection only at leading order in $M\delta t$. The determination of the event horizon of the Vaidya metric at higher orders in $M\delta t$, is an academic exercise that we solve in this subsection largely because it illustrates the procedure one could adopt on the full metric.

We then plug these expansions into the equations of motion, expand these equations in a power series in ϵ, and proceed to solve these equations recursively, order by order in ϵ.

The formal structure of this procedure is familiar. The coefficient of ϵ^n in the equations of motion take the schematic form

$$H^i_j \chi^j_n(r,v)) = s^i_n \tag{2.33}$$

Here χ^i_N stands for the three dimensional 'vector' of n^{th} order unknowns, i.e. $\chi^1_n = f_n$, $\chi^2_n = g_n$ and $\chi^3_n = \phi_n$. The differential operator H^i_j is universal (in the sense that it is the same at all n) and has a simple interpretation; it is simply the operator that describes linearized fluctuations about AdS space. The source functions s^i_n are linear combinations of products of χ^i_m $(m < n)$; the sum over m over fields that appear in any particular term adds up to n.

The equations (2.33) are to be solved subject to the large r boundary conditions

$$\begin{aligned} \lim_{r\to\infty} \phi_1(r,v) &= \phi_0(r) \\ \phi_n(r,v) &\leq \mathcal{O}(1/r), \quad n \geq 2 \\ f_n(r,v) &\leq \mathcal{O}(1/r), \quad n \geq 1 \\ g_n(r,v) &\leq \mathcal{O}(r), \qquad n \geq 1 \end{aligned} \tag{2.34}$$

together with the initial conditions

$$\phi_n(r,v) = g_n(r,v) = f_n(r,v) = 0 \quad \text{for} \quad v < 0 \quad (n \geq 1) \tag{2.35}$$

These boundary and initial conditions uniquely determine ϕ_n, g_n and f_n in terms of the source functions.

All sources vanish at first order in perturbation theory (i.e the functions s^i_1 are zero). Consequently, the functions f_1 and g_1 vanish but ϕ_1 is forced by its boundary condition to be nonzero. As we will see below, it is easy to explicitly solve for the function ϕ_1. This solution, in turn, completely determines the source functions at $\mathcal{O}(\epsilon^2)$ and so the equations (2.33) unambiguously determine g_2, ϕ_2 and f_2. This story repeats recursively. The solution to perturbation theory at order $n-1$ determine the source functions at order n and so permits the determination of the unknown functions at order n. The final answer, at every order, is uniquely determined in terms of $\phi_0(v)$.

To end this subsection, we note a simplifying aspect of our perturbation theory. It follows from the structure of the equations that ϕ_n is nonzero only when n is odd while f_m and g_m are nonzero only when m is even. We will use this fact extensively below.

2.6 Explicit results for naive perturbation theory to fifth order

We have implemented the naive perturbative procedure described above to $\mathcal{O}(\epsilon^5)$. Before proceeding to a more structural discussion of the nature of the perturbative expansion, we pause here to record our explicit results.

At leading (first and second) order we find

$$
\begin{aligned}
\phi_1(r,v) &= \phi_0(v) + \frac{\dot{\phi}_0}{r} \\
f_2(r,v) &= -\frac{\dot{\phi}_0^2}{8r} \\
g_2(r,v) &= -\frac{C_2(v)}{r} - \frac{3}{4}\dot{\phi}_0^2
\end{aligned}
\tag{2.36}
$$

At the next order

$$
\begin{aligned}
\phi_3(r,v) &= \frac{1}{4r^3}\int_{-\infty}^{v} B(x)\ dx \\
f_4(r,v) &= \frac{\dot{\phi}_0}{384r^3}\left\{\dot{\phi}_0^3 - 12\int_{-\infty}^{v} B(x)\ dx\right\} \\
g_4(r,v) &= \frac{C_4(v)}{r} + \frac{\dot{\phi}_0}{24r^2}\left\{-\dot{\phi}_0^3 + 3\int_{-\infty}^{v} B(x)\ dx\right\} \\
&\quad + \frac{1}{48r^3}\left(3B(v)\dot{\phi}_0 - 4\dot{\phi}_0^3\ddot{\phi}_0 + 3\ddot{\phi}_0\int_{v}^{\infty} B(t)dt\right)
\end{aligned}
\tag{2.37}
$$

while ϕ_5 is given by

$$
\begin{aligned}
\phi_5(r,v) = \frac{1}{8r^5}&\int_{-\infty}^{v} B_1(x)\ dx \\
&+ \frac{1}{6r^4}\int_{-\infty}^{v} B_3(x)\ dx + \frac{5}{24r^4}\int_{-\infty}^{v} dy\int_{-\infty}^{y} B_1(x)\ dx \\
&+ \frac{1}{4r^3}\int_{-\infty}^{v} B_2(x)\ dx + \frac{1}{6r^3}\int_{-\infty}^{v} dy\int_{-\infty}^{y} B_3(x)\ dx \\
&+ \frac{5}{24r^3}\int_{-\infty}^{v} dz\int_{-\infty}^{z} dy\int_{-\infty}^{y} B_1(x)\ dx
\end{aligned}
\tag{2.38}
$$

In the equations above

$$
\begin{aligned}
B(v) &= \dot{\phi}_0\left[-C_2(v) + \dot{\phi}_0\ddot{\phi}_0\right] \\
B_1(v) &= \left(-\frac{9}{4}C_2(v) + \frac{7}{8}\dot{\phi}_0\ddot{\phi}_0\right)\int_{-\infty}^{v} B(x)\ dx \\
&\quad + \frac{1}{2}C_2(v)\dot{\phi}_0^3 + \frac{3}{8}\dot{\phi}_0^2 B(v) - \frac{1}{6}\dot{\phi}_0^4\ddot{\phi}_0 \\
B_2(v) &= C_4(v)\dot{\phi}_0 \\
B_3(v) &= \frac{1}{24}\left(-30\dot{\phi}_0^2\int_{-\infty}^{v} B(x)\ dx + 7\dot{\phi}_0^5\right)
\end{aligned}
\tag{2.39}
$$

and the energy functions $C_2(v)$ and $C_4(v)$ (obtained by integrating the energy conservation equation) are given by

$$
\begin{aligned}
C_2(v) &= -\int_{-\infty}^{v} dt\frac{1}{2}\dot{\phi}_0\dddot{\phi}_0 \\
C_4(v) &= \int_{-\infty}^{v} dt\frac{3}{8}\dot{\phi}_0\left(-\dot{\phi}_0^3 + \int_{-\infty}^{t} B(x)\ dx\right)
\end{aligned}
\tag{2.40}
$$

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For use below, we note in particular that at $v = \delta t$ the mass of the black brane is given by $C_2(\delta t) - C_4(\delta t) + \mathcal{O}(\epsilon^6)$ while the value of the dilaton field is given by

$$\begin{aligned}
\phi(r, \delta t) = &\frac{1}{4r^3}\int_{-\infty}^{\delta t} B(x)\, dx \\
&+ \frac{1}{4r^3}\int_{-\infty}^{\delta t} B_2(x)\, dx + \frac{1}{6r^3}\int_{-\infty}^{\delta t} dy \int_{-\infty}^{y} B_3(x)\, dx \\
&+ \frac{5}{24r^3}\int_{-\infty}^{\delta t} dz \int_{-\infty}^{z} dy \int_{-\infty}^{y} B_1(x)\, dx \\
&+ \frac{5}{24r^4}\int_{-\infty}^{\delta t} dy \int_{-\infty}^{y} B_1(x)\, dx + \frac{1}{6r^4}\int_{-\infty}^{\delta t} B_3(x)\, dx \\
&+ \frac{1}{8r^5}\int_{-\infty}^{\delta t} B_1(x)\, dx \quad + \mathcal{O}(\epsilon^7)
\end{aligned} \tag{2.41}$$

2.7 The analytic structure of the naive perturbative expansion

In this subsection we will explore the analytic structure of the naive perturbation expansion in the variables v (for $v > \delta t$) and r. It is possible to inductively demonstrate that

1. The functions ϕ_{2n+1}, g_{2n+2} and f_{2n+2} have the following analytic structure in the variable r

$$\begin{aligned}
\phi_{2n+1}(r, v) &= \sum_{k=0}^{2n-2} \frac{\phi^k_{2n+1}(v)}{r^{2n+1-k}}, && (n \geq 2) \\
f_{2n}(r, v) &= r \sum_{k=0}^{2n-6} \frac{f^k_{2n}(v)}{r^{2n-k}}, && (n \geq 3) \\
g_{2n}(r, v) &= \frac{C_{2n}(\delta t)}{r} + r \sum_{k=0}^{2n-5} \frac{g^k_{2n-3}(v)}{r^{2n-k}}, && (n \geq 3)
\end{aligned} \tag{2.42}$$

   Moreover, when $v > \delta t$ $\phi_1(r, v) = f_2(r, v) = f_4(r, v) = 0$ while $g_2(r, v) = -\frac{C_2(\delta t)}{r}$ and $g_4(r, v) = \frac{C_4(\delta t)}{r}$.

2. The functions $\phi^k_{2n+1}(v)$, $f^k_{2n}(v)$ and $g^k_{2n}(v)$ are each functionals of $\phi_0(v)$ that scale like $\lambda^{-2n-1+k}$, λ^{-2n+k} and $\lambda^{-2n+k-1}$ respectively under the scaling $v \to \lambda v$.

3. For $v > \delta t$ the functions $\phi^k_{2n+1}(v)$ are all polynomials in v of a degree that grows with n. In particular the degree of ϕ^k_{2n+1} at most $n - 1 + k$; the degree of f^k_{2n} is at most $n - 3 + k$ and the degree of g^k_{2n} is at most $n - 4 + k$.

The reader may easily verify that all these properties hold for the explicit low order solutions of the previous subsection.

2.8 Infrared divergences and their cure

The fact that $\phi_{2n+1}(v)$ are polynomials in time whose degree grows with n immediately implies that the naive perturbation theory of the previous subsection fails at late positive times. We pause to characterize this failure in more detail. As we have explained above, the field $\phi(r,v)$ schematically takes the form

$$\sum_{n,k} \frac{\epsilon^{2n+1}\phi^k_{2n+1}}{r^{2n+1-k}}$$

where $\phi^k_{2n+1} \sim \frac{v^{n-1+k}}{(\delta t)^{3n}}$ at large times. Let us examine this sum in the vicinity $r \sim \frac{\epsilon^{\frac{2}{3}}}{\delta t}$, a surface that will turn out to be the event horizon of our solution. The term with labels n,k scales like $\epsilon \times (\epsilon^{\frac{2}{3}} \frac{v}{\delta t})^{n-1+k}$. Now $\frac{\epsilon^{\frac{2}{3}}}{\delta t} = T$ is approximately the temperature of a black brane of event horizon r_H. We conclude that the term with labels n,k scales like $(vT)^{n-1+k}$. It follows that, at least in the vicinity of the horizon, the naive expansion for ϕ is dominated by the smallest values of n and k when $\delta t T \ll 1$. On the other hand, at times large compared to the inverse temperature, this sum is dominated by the largest values of k and n. As the sum over n runs to infinity, it follows that naive perturbation theory breaks down at time scales of order T^{-1}.

A long time or IR divergence in perturbation theory usually signals the fact that the perturbation expansion has been carried out about the wrong expansion point; i.e. the zero order 'guess' with which we started perturbation (empty AdS space) does not everywhere approximate the true solution even at arbitrarily small ϵ. Recall that naive perturbation theory is perfectly satisfactory for times of order δv so long as $r \gg \frac{\epsilon}{\delta t}$. Consequently this perturbation theory may be used to check if our spacetime metric deviates significantly from the pure AdS in this range of r and at these early times. The answer is that it does, even in the limit $\epsilon \to 0$. In order to see precisely how this comes about, note that the most singular term in g_{2n} is of order $r \times \frac{1}{r^{2n}}$ for $n \geq 1$, the exact value of $g_0 = r^2 = (r \times \frac{1}{r^0} \times r)$. In other words g_0 happens to be less singular, near $r = 0$, than one would expect from an extrapolation of the singularity structure of g_n at finite n down to $n = 0$. As a consequence, even though g_0 is of lowest order in ϵ, at small enough r it is dominated by the most singular term in $g_2(r,v)$. Moreover this crossover in dominance occurs at $r \sim \frac{\epsilon^{\frac{2}{3}}}{\delta t} \gg \frac{\epsilon}{\delta t}$ and so occurs well within the domain of applicability of perturbation theory. In other words, in the variable range $r \gg \frac{\epsilon}{\delta t}$, $g(r,v)$ is not uniformly well approximated by $g_0 = r^2$ at small ϵ but instead by

$$g(r,v) \approx r^2 - \frac{C_2(v)}{r}.$$

This implies that, in the appropriate parameter range, the true metric of the spacetime is everywhere well approximated by the Vaidya metric (1.1), with $M(v)$ given by (2.20) in the limit $\epsilon \to 0$.

Of course even this corrected estimate for $g(r,v)$ breaks down at $r \sim \frac{\epsilon}{\delta t}$. However, as we have indicated above, this will turn out to be irrelevant for our purposes as our spacetime develops an event horizon at $r \sim \frac{\epsilon^{\frac{2}{3}}}{\delta t}$.

We will now proceed to argue that the metric is well approximated by the Vaidya form at all times (not just at early times) outside its event horizon, so that the Vaidya metric (1.1) rather than empty AdS space, constitutes the correct starting point for the perturbative expansion of our solution.

2.9 The metric to leading order at all times

The dilaton field and spacetime metric begin a new stage in their evolution at $v = \delta t$. At later times the solution is a normalizable, asymptotically AdS solution to the equations of motion. This late time motion is unforced and so is completely determined by two pieces of initial data; the mass density $M(\delta t)$ and the dilaton function $\phi(r, \delta t)$. As the naive perturbation expansion described in subsection 2.7 is valid at times of order δt, it determines both these quantities perturbatively in ϵ. The explicit results for these quantities, to first two nontrivial orders in ϵ, are listed in (2.41).

The leading order expression for the mass density is simply given by C_2 in (2.20). Now if one could ignore $\phi(r, \delta t)$ (i.e. if this function were zero) this initial condition would define a unique, simple subsequent solution to Einstein's equations; the uniform black brane with mass density C_2. While $\phi(r, \delta t)$ is not zero, we will now show it induces only a small perturbation about the black brane background.

In order to see this it is useful to move to a rescaled variable $\tilde{r} = \frac{r}{C_2^{\frac{1}{3}}}$. In terms of this rescaled variable, our solution at $v = \delta t$ is a black brane of unit energy density, perturbed by $\phi(r, \delta t)$. With this choice of variable the background metric is independent of ϵ, so that all ϵ dependence in our problem lies in the perturbation. It follows that, to leading order in ϵ (recall $\phi_1(r, \delta t) = 0$)

$$\phi(r, \delta t) = \frac{\phi_3^0(\delta t)}{r^3}\left(1 + \mathcal{O}(\epsilon^{\frac{2}{3}})\right) = \frac{1}{\tilde{r}^3} \times \frac{\phi_3^0(\delta t)}{M}\left(1 + \mathcal{O}(\epsilon^{\frac{2}{3}})\right) \sim \frac{\epsilon}{\tilde{r}^3} \tag{2.43}$$

where, from subsection 2.6

$$\phi_3^0(\delta t) = \frac{1}{4}\int_{-\infty}^{\delta t} B(x)\, dx \tag{2.44}$$

The important point here is that the perturbation is proportional to ϵ and so represents a small deformation of the dilaton field about the unit energy density black brane initial condition. Moreover, any regular linearized perturbation about the black brane may be re expressed as a linear sum of quasinormal modes about the black brane and so decays exponentially over a time scale of order the inverse temperature. It follows that the initialy small dilaton perturbation remains small at all future times and in fact decays exponentially to zero over a finite time. The fact that perturbations about the Vaidya metric (1.1) are bounded both in amplitude as well as in temporal duration allows us to conclude that the event horizon of the true spacetime is well approximated by the event horizon of the Vaidya metric at small ϵ, as described in subsection 2.4.

2.10 Resummed versus naive perturbation theory

Let us define a resummed perturbation theory which uses the corrected metric (1.1) (rather than the unperturbed AdS metric) as the starting point of an amplitude expansion. This amounts to correcting the naive perturbative expansion by working to all orders in $M \sim \epsilon^2$, while working perturbatively in all other sources of ϵ dependence.[18] As we have argued above, resummed perturbation theory (unlike its naive counterpart) is valid at all times.

[18]This is conceptually similar to the coupling constant expansion in finite temperature weak coupling QED. There, as in our situation, naive perturbation theory leads to IR divergences, which are cured upon

We have seen above that the naive perturbation theory gives reliable results when $vT \ll 1$. This fact has a simple 'explanation'; we will now argue that the resummed perturbation theory (which is always reliable at small ϵ) agrees qualitatively with naive perturbation theory $vT \ll 1$.

At each order, resummed perturbation theory involves solving the equation

$$\partial_r \left[r^4 \left(1 - \frac{M(v)}{r^3} \right) \partial_r \phi \right] + 2r\partial_v \partial_r (r\phi) = \text{source} \tag{2.45}$$

The naive perturbation procedure requires us to solve an equation of the same form but with M set to zero. In the vicinity of the horizon, the two terms in the expression $(1 - \frac{M(v)}{r^3})$ are comparable, so that the resummed and naive perturbative expansions can agree only when the entire first term on the l.h.s. of (2.45) is negligible compared to the second term on the l.h.s. of the same equation. The ratio of the first term to the second may be approximated by rv where v is the time scale for the process in question. Now the term multiplying the mass in (2.45) is only important in the neighborhood of the horizon, where $r \sim M^{\frac{1}{3}} \sim T$ where T is the temperature of the black brane. It follows that resummed and naive perturbation expansions will differ substantially from each other only at time scales of order and larger than the inverse temperature.

Let us restate the point in a less technical manner. The evolution of a field ϕ, outside the horizon of a black brane of temperature T, is not very different from the evolution of the same field in Poincare patch AdS space, over time scales v where $vT \ll 1$. However the two motions differ significantly over time scales of order or greater than the inverse temperature. In particular, in the background of the black brane, the field ϕ outside the horizon decays exponentially with time over a time scale set by the inverse temperature; i.e. the solution involves factors like e^{-vT}. As the temperature is itself of order $\epsilon^{\frac{2}{3}}$, naive perturbation theory deals with these exponentials by power expanding them. Truncating to any finite order then gives apparently divergent behavior at large times. Resummed perturbation theory makes it apparent that these divergences actually resum into completely convergent, decaying, exponentials.

2.11 Resummed perturbation theory at third order

In the previous subsection we have presented explicit results for the behavior of the dilaton and metric fields, at small ϵ and for early times $vM^{\frac{1}{3}} \ll 1$. The resummed perturbation theory outlined in this section may be used to systematically correct the leading order spacetime (1.1) at all times, in a power series in $\epsilon^{\frac{2}{3}}$. In this section we explicitly evaluate the leading order correction in terms of a universal (i.e. ϕ_0 independent) function $\psi(x, y)$, whose explicit form we are able to determine only numerically.

Let us define the function $\psi(x, y)$ as the unique solution of the differential equation

$$\partial_x \left(x^4 \left(1 - \frac{1}{x^3} \right) \partial_x \psi \right) + 2x\partial_y \partial_x \left(x\psi \right) = 0 \tag{2.46}$$

exactly accounting for the photon mass (which is of order $g_{\rm YM}^2$). Resummed perturbation theory in that context corresponds to working with a modified propagator which effectively includes all order effects in the photon mass, while working perturbatively in all other sources of the fine structure constant α.

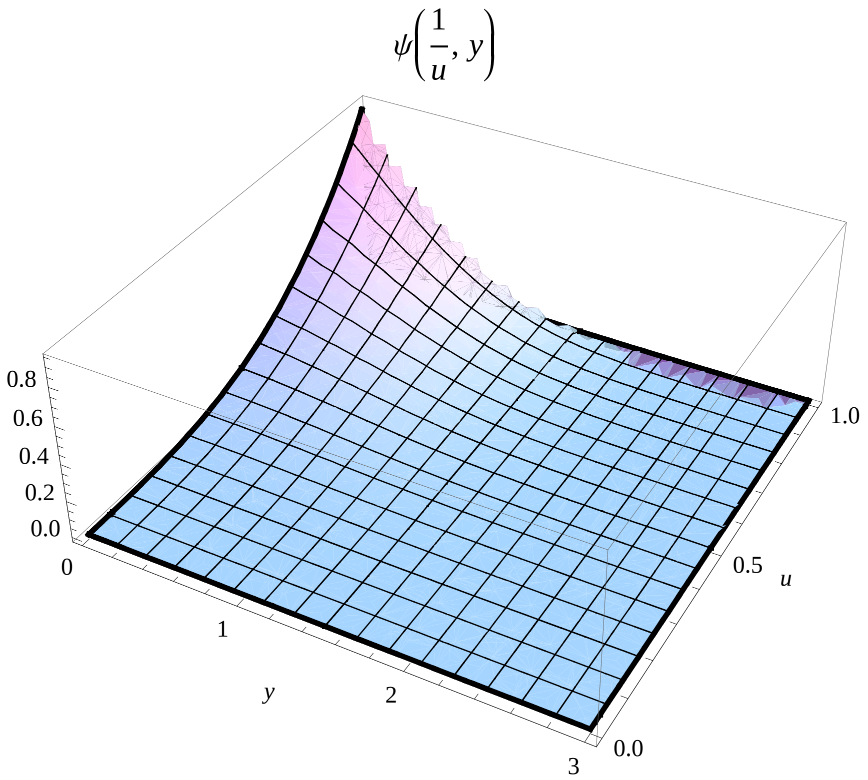

Figure 3. Numerical solution for dilaton to the leading order in amplitude at late time

subject to the boundary condition $\psi \sim \mathcal{O}(\frac{1}{x^3})$ at large x and the initial condition $\psi(x,0) = \frac{1}{x^3}$. The leading order solution to the resummed perturbation theory for ϕ, for $v > \delta t$, is given by

$$\phi = \frac{\phi_3^0(\delta t)}{M} \psi\left(\frac{r}{M^{\frac{1}{3}}}, (v - \delta t) M^{\frac{1}{3}}\right) \tag{2.47}$$

Unfortunately, the linear differential equation (2.46) — appears to be difficult to solve analytically. In this section we present a numerical solution of (2.46). Although we are forced to resort to numerics to determine $\psi(x,y)$, we emphasize that a single numerical evaluation suffices to determine the leading order solution at all values of the forcing function $\phi_0(v)$. This may be contrasted with an ab initio numerical approach to the full nonlinear differential equations, which require the re running of the full numerical code for every initial function ϕ_0. In particular the ab initio numerical method cannot be used to prove general statements about a wide class of forcing functions ϕ_0.

In figure 3 we present a plot of $\psi(\frac{1}{u}, y)$ against the variables u and y. The exterior of the event horizon lives in the compact interval $\frac{1}{x} = u \in (0,1)$, and in our figure y runs from zero to three.

In order to obtain this plot we rewrote the differential equation (2.46) in terms of the variable $u = \frac{1}{x}$ (as explained above) and worked with the field variable $\chi(u,y) = (1-u)\psi(\frac{1}{u}, y)$. Recall that our original field ψ is expected to be regular at the horizon $u = 1$ at all times. This expectation imposes the boundary condition $\chi(.999999, y) = 0$.

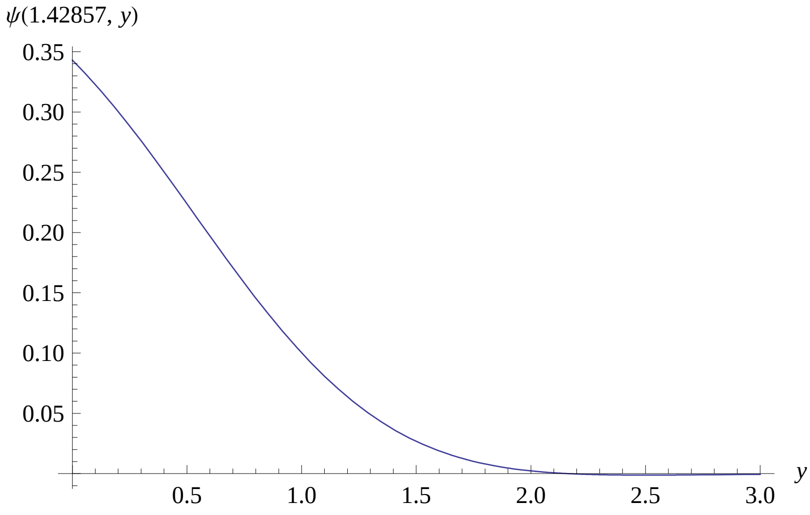

Figure 4. A plot of $\psi(\frac{1}{0.7}, y)$ as a function of y

We further imposed the condition of normalizability $\chi(0, y) = 0$ and the initial condition $\chi(u, 0) = (0.999999 - u)u^3$. Of course 0.999999 above is simply a good approximation to 1 that avoids numerical difficulties at unity. The partial differential equation solving routine of Mathematica-6 was able to solve our equation subject to these boundary and initial conditions, with a step size of 0.0005 and an accuracy goal of 0.001; we have displayed this Mathematica output in figure 3. In order to give a better feeling for the function $\psi(x, y)$ in figure 4 we present a graph of $\psi(\frac{1}{0.7}, y)$ (i.e. as a function of time at a fixed radial location). Notice that this graph decays, roughly exponentially for $v > 0.5$ and that this exponential decay is dressed with a sinusodial osciallation, as expected for quasinormal type behavior. A very very rough estimate of this decay constant ω_I may be obtained from equation $\frac{\psi(\frac{1}{0.7},1.5)}{\psi(\frac{1}{0.7},.5)} = e^{-\omega_I}$ which gives $\omega_I \approx 8.9T$ (here T is the temperature of our black brane given by $T = \frac{4\pi}{3}$). This number is the same ballpark as the decay constant for the first quasi normal mode of the uniform black brane, $\omega_I = 11.16T$, quoted in [40].

3 Spherically symmetric asymptotically flat collapse

3.1 The set up

In this section[19] we study spherically symmetric asymptotically flat solutions to Einstein gravity (with no cosmological constant) interacting with a minimally coupled massless scalar field, in 4 bulk dimensions. The Lagrangian for our system is

$$S = \int d^4x\sqrt{g}\left(R - \frac{1}{2}(\partial\phi)^2\right) \tag{3.1}$$

[19]We thank B. Kol and O. Aharony for discussions that led us to separately study collapse in flat space.

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We choose a gauge so that our metric and dilaton take the form

$$\begin{aligned} ds^2 &= 2drdv - g(r,v)dv^2 + f^2(r,v)d\Omega_2^2 \\ \phi &= \phi(r,v). \end{aligned} \tag{3.2}$$

where $d\Omega_2^2$ is the line element on a unit two sphere. We will explore solutions to the equations of motion of this system subject to the pure flat space initial conditions

$$\begin{aligned} g(r,v) &= 1, \quad (v<0) \\ f(r,v) &= r, \quad (v<0) \\ \phi(r,v) &= 0, \quad (v<0) \end{aligned} \tag{3.3}$$

and the large r boundary conditions

$$\begin{aligned} g(r,v) &= 1 + \mathcal{O}\left(\frac{1}{r}\right) \\ f(r,v) &= r\left(1 + \mathcal{O}\left(\frac{1}{r^2}\right)\right) \\ \phi(r,v) &= \frac{\psi(v)}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \end{aligned} \tag{3.4}$$

where $\psi(v)$ takes the form

$$\begin{aligned} \psi(v) &= 0, \qquad (v<0) \\ \psi(v) &< \epsilon_f \delta t, \quad (0<v<\delta t) \\ \psi(v) &= 0 \qquad (v>\delta t), \end{aligned} \tag{3.5}$$

In other words our spacetime starts out in its vacuum, but has a massless pulse of limited duration focused to converge at the origin at $v=0$. This pulse could lead to interesting behavior — like black hole formation, as we explore in this section.

The structure of the equations of motion of our system was described in subsection 2.2. As in that subsection, the independent dynamical equations for our system may be chosen to be the dilaton equation of motion plus the two constraint equations, supplemented by an energy conservation equation. The explicit form of the dilaton and constraint equations is given by

$$\begin{aligned} \partial_r\left(f^2 g\partial_r\phi\right) + \partial_v\left(f^2\partial_r\phi\right) + \partial_r\left(f^2\partial_v\phi\right) &= 0 \\ (\partial_r\phi)^2 &= -\frac{4\partial_r^2 f}{f} \\ \partial_r\left(fg\partial_r f + 2f\partial_v f\right) &= 1 \end{aligned} \tag{3.6}$$

As in the previous section, we may choose to evaluate the energy conservation equation at large r. As we have explained, the large r behavior of the function g is given by

$$g(r,v) = 1 - \frac{M(v)}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{3.7}$$

The energy conservation equation, evaluated at large r, yields

$$\dot{M} = -\frac{\psi \dddot{\psi}}{2} \tag{3.8}$$

The equations(3.6) together with (3.8) constitute the full set of dynamical equations for our problem.

By integrating (3.8) we find an exact expression for $M(v)$

$$M(v) = \frac{-\psi \ddot{\psi} + \int_{-\infty}^{v} \ddot{\psi}^2}{2} \tag{3.9}$$

Note in particular that $M(v)$ reduces to a constant M for $v > \delta t$ where

$$M = \frac{\int_{-\infty}^{\delta t} \ddot{\psi}^2}{2} \sim \epsilon_f^2 \delta t \tag{3.10}$$

3.2 Regular amplitude expansion

Our equations may be solved in the amplitude expansion formally described in (2.5), i.e. in an expansion in powers of the function $\psi(v)$. As we will argue in this paper, there are two inequivalent valid amplitude expansions of these equations. In the first, the spacetime is everywhere regular and the dilaton is everywhere small. In the second, the spacetime is singular at small r but this singularity is shielded from asymptotic infinity by a regular event horizon. The second amplitude expansion reliably describes the spacetime only outside the event horizon; this expansion works because the dilaton is uniformly small outside the event horizon. As we will see two amplitude expansions described above have non overlapping regimes of validity, and so describe dynamics in different regimes of parameter space.

In this subsection we briefly comment on the more straightforward fully regular expansion. At every order in perturbation theory, the requirement or regularity uniquely determines the solution. Explicitly at first order we have

$$\phi_1(r, v) = \frac{\psi(v) - \psi(v - 2r)}{r} \tag{3.11}$$

The perturbation expansion that starts with this solution is valid only when $\phi(r)$ is everywhere small. $\phi(r)$ reaches its maximum value near the origin, and $\phi_1(0, v) \sim 2\dot{\psi}(v) \sim \epsilon_f$. Consequently the regular perturbation expansion, sketched in this section, is valid only when $\epsilon_f \ll 1$ i.e. when $\frac{\delta t}{M} \gg 1$.

At next order in the amplitude expansion we find

$$\begin{aligned} f_2(r, v) &= \frac{1}{4} \left(r \int_r^\infty \rho \left[\partial_\rho \phi_1(\rho, v)\right]^2 \, d\rho - \int_r^\infty \rho^2 \left[\partial_\rho \phi_1(\rho, v)\right]^2 \, d\rho \right) \\ g_2(r, v) &= -2\partial_v f_2(r, v) - \frac{f_2(r, v) - f_2(0, v)}{r} - \partial_r f_2(r, v) \end{aligned} \tag{3.12}$$

The integration limits in the expression for $f_2(r, v)$ in 3.12 are fixed such that at large r $f(r, v)$ decays like $\frac{1}{r}$. The integration constant in $g_2(r, v)$ is fixed by the requirement that the solution be regular at $r = 0$.

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3.2.1 Regularity implies energy conservation

In this subsection we pause to explain an interesting technical subtlety that arises in carrying out the regular amplitude expansion. The discussion of this subsection will play no role in the analysis of spacetimes that describe black hole formation, so the reader who happens to be uninterested in the regular expansion could skip to the next section.

Note that in order to obtain (3.12) we did not make any use of the energy conservation equation. We will now verify (first in terms of the answer, and then more abstractly) that (3.12) automatically obeys the energy conservation equation. At large r, these functions have the following expansion

$$\begin{aligned} \phi_1(r,v) &= \frac{\psi(v)}{r} \\ f_2(r,v) &= -\frac{\psi(v)^2}{8r} \\ g_2(r,v) &= -\frac{C_2(v)}{r}, \quad \text{where} \\ C_2(v) &= -\frac{\psi(v)\dot{\psi}(v)}{2} - f_2(0,v) \end{aligned} \tag{3.13}$$

If our solution does indeed obey the energy conservation relation, we should find that $C_2(v)$ is equal to $M(v)$ in (3.10). We will now proceed to directly verify that this is the case.

The first term in $C_2(v)$ comes from the coefficient of $\frac{1}{r}$ in $\partial_v f_2(r,v)$. For the second term in the expression for $C_2(v)$, $f_2(0)$, is given by

$$f_2(0,v) = -\frac{1}{4}\int_0^\infty \rho^2 \left[\partial_\rho \phi_1(\rho,v)\right]^2 \, d\rho$$

The integrand in this expression may be split into four terms in the following way.

$$\begin{aligned} r^2\left[\partial_r\phi_1(r,v)\right]^2 &= 2\psi(v)\partial_r\left[\frac{\psi(v-2r)}{r}\right] + \frac{\psi(v)^2}{r^2} + r^2\left[\partial_r\left(\frac{\psi(v-2r)}{r}\right)\right]^2 \\ &= 2\psi(v)\partial_r\left[\frac{\psi(v-2r)}{r}\right] + \frac{\psi(v)^2}{r^2} + 4\left[\dot{\psi}(v-2r)\right]^2 - \partial_r\left[\frac{\psi^2(v-2r)}{r}\right] \end{aligned} \tag{3.14}$$

Now each of the terms can be integrated.

$$\begin{aligned} \int_0^r 2\psi(v)\partial_\rho\left[\frac{\psi(v-2\rho)}{\rho}\right] d\rho = -2\lim_{r\to 0}\frac{\psi(v)\psi(v-2r)}{r} &= -2\lim_{r\to 0}\frac{\psi(v)^2}{r} \\ \int_0^r \frac{\psi(v)^2}{\rho^2} d\rho &= \lim_{r\to 0}\frac{\psi(v)^2}{r} \\ \int_0^r 4\left[\dot{\psi}(v-2\rho)\right]^2 d\rho &= 2\int_{-\infty}^v \dot{\psi}(t)^2 \, dt \\ -\int_0^r \partial_\rho\left[\frac{\psi^2(v-2\rho)}{\rho}\right] d\rho = \lim_{r\to 0}\frac{\psi(v-2r)^2}{r} &= \lim_{r\to 0}\frac{\psi(v)^2}{r} \end{aligned} \tag{3.15}$$

Adding all the terms one finally finds

$$-f_2(0,v) = \frac{1}{2}\int_{-\infty}^v \dot{\psi}(t)^2 \, dt \tag{3.16}$$

This implies

$$C_2(v) = -\frac{\psi(v)\dot{\psi}(v)}{2} + \frac{1}{2}\int_{-\infty}^{v} \dot{\psi}(t)^2 \, dt = M(v) \tag{3.17}$$

as expected from energy conservation.

Let us summarize In order to obtain our result for g_2 above, we were required to fix the value of an integration constant. The value of this constant may determined in two equally valid ways

- By imposing the energy conservation equation $E_{\rm ec}$
- By demanding regularity of the solution at $r = 0$

In fact these two conditions are secretly the same, as we now argue. As we have explained in subsection 2.2, $\partial_r(rE_{\rm ec})$ automatically vanishes whenever the three equations (3.6) are obeyed. Consequently, if $rE_{\rm ec}$ vanishes at any one value of r it automatically vanishes at every r. Now the equation $E_{\rm ec}$ evaluates to a finite value at $r = 0$ provided our solution is regular at $r = 0$. It follows that the regular solution automatically has $rE_{\rm ec} = 0$ everywhere.

Configurations in the amplitude expansion of the previous section (or the singular amplitude expansion we will describe shortly below), on the other hand, are all singular at $r = 0$. $rE_{\rm ec}$ does not automatically vanish on these solutions, and the energy conservation equation $E_{\rm ec}$ is not automatic but must be imposed as an additional constraint on solutions.

It would be a straightforward — if cumbersome — exercise to explicitly implement the perturbation theory, described in this subsection, to higher orders in ϵ_f. As our main interest is black hole formation, we do not pause to do that.

3.3 Leading order metric and event horizon for black hole formation

In the rest of this section we will describe the formation of black holes in flat space in an amplitude expansion. In contrast with the previous subsection, our amplitude expansion will be justified by the small parameter $\frac{1}{\epsilon_f}$. Our analysis will reveal that our spacetime takes the Vaidya form to leading order in $\frac{1}{\epsilon_f^2}$,

$$ds^2 = 2drdv - \left(1 - \frac{M(v)}{r}\right) dv^2 + r^2 d\Omega_2^2 \tag{3.18}$$

where $M(v)$ is given by (3.9).

In this subsection we will compute the event horizon of the spacetime (3.18) at large ϵ_f. We present the computation of this event horizon even before we have justified the form (3.18), as our aim in subsequent subsections is to have a good perturbative expansion of the true solution only outside the event horizon; consequently the results of this subsection will guide the construction of the amplitude expansion in subsequent subsections.

As in the previous section the event horizon takes the form

$$\begin{aligned} r_H(v) &= M, && (v > \delta t) \\ r_H(v) &= Mx\left(\frac{v}{\delta t}\right), && (0 < v < \delta t) \\ r_H(v) &= Mx(0) + v, && (-x(0) < v < 0) \end{aligned} \tag{3.19}$$

where the function $x(t)$ may easily be evaluated in a power series in $\frac{\delta t}{M} \sim \frac{1}{\epsilon_f^2}$. We find

$$\begin{aligned} x(t) &= 1 + \left(\frac{\delta t}{M}\right) x_1(t) + \left(\frac{\delta t}{M}\right)^2 x_2(t) + \dots \\ x_1(t) &= -\int_t^1 dy \left(\frac{1 - \frac{M(y\delta t)}{M}}{2}\right) \\ x_2(t) &= -\int_t^1 dy x_1(y) \frac{M(y\delta t)}{M}. \end{aligned} \tag{3.20}$$

In particular $r_H = M$ for all $v > 0$ at leading order.

3.4 Amplitude expansion for black hole formation

Let us now construct an amplitude expansion (i.e. expansion in powers of $\psi(v)$) of our solution in the opposite limit to that of the previous subsection, namely $\frac{M}{\delta t} \sim \epsilon_f^2 \gg 1$. It is intuitively clear that such a dilaton shell will propagate into its own Schwarzschild radius and then cannot expand back out to infinity. In other words the second term in (3.11) cannot form a good approximation to the leading order solution for the collapse of such a shell. Now (3.11) deviates from

$$\phi_1(r, v) = \frac{\psi(v)}{r}; \tag{3.21}$$

only at spacetime points that feel the back scattered expanding wave in (3.11). This observation suggests that (3.21) itself is the appropriate starting point for the amplitude expansion at large ϵ_f, and this is indeed the case.

The incident dilaton pulse (3.21) will back react on the metric; above we have derived an exact expression for one term — roughly the Newtonian potential — (see (3.10)) of this back reacted metric. Including this backreaction (all others turn out to be negligible at large ϵ_f) the spacetime metric takes the form

$$ds^2 = 2dvdr - dv^2(1 - \frac{M(v)}{r}) + r^2 d\Omega_2^2 \tag{3.22}$$

As we have explained in the previous subsection, this solution has an event horizon located at $r_H \sim M \sim \epsilon_f^2 \delta t$ for $v > 0$ (see below). Consequently, $\phi_1(r, v)$ outside the event horizon $\leq \frac{\psi}{r_H} \sim \frac{1}{\epsilon_f} \sim \sqrt{\frac{\delta t}{r_H}}$, i.e. is parametrically small at large ϵ_f. This fact allows us to construct a large ϵ_f amplitude expansion for the solution outside its event horizon.

The perturbation expansion of our solutions in $\frac{\delta t}{M}$ is similar in many ways to the perturbation theory described in detail in section 2. As in that section, the true (resummed) expansion (built around the starting metric (3.22)) is well approximated at early times by a naive expansion built around unperturbed flat space. Naive and resummed expansions agree whenever the first term in the first equation of (3.6) is negligible compared to the other terms in that equation, i.e. for $v \ll M \sim \epsilon_f^2 \delta t$. As ϵ_f is large in this subsection, naive and resummed perturbation theory are simultaneously valid for times that are of order δt. However we expect the naive expansion to break down at $v \gg M$. We will now study the naive expansion in more detail and confirm these expectations.

3.5 Analytic structure of the naive perturbation expansion

In this subsection we describe the structure of a perturbative expansion built starting from the flat space metric. We expand the full solution as

$$\begin{aligned}\phi(r,v) &= \sum_{n=0}^{\infty}\Phi_{2n+1}\\ f(r,v) &= r+\sum_{n=1}^{\infty}F_{2n}(r,v)\\ g(r,v) &= 1+\sum_{n=1}^{\infty}G_{2n}(r,v)\end{aligned} \tag{3.23}$$

where, by definition, the functions Φ_m F_m and G_m are each of homogeneity m in the source function $\psi(v)$. As explained above we take

$$\Phi_1(r,v) = \frac{\psi(v)}{r} \tag{3.24}$$

By studying the formal structure of the perturbation expansion, it is not difficult to inductively establish that

1. The functions Φ_{2n+1}, F_{2n} and G_{2n} have the following analytic structure in the variable r

$$\begin{aligned}\Phi_{2n+1}(r,v) &= \sum_{m=0}^{\infty}\frac{\Phi_{2n+1}^{m}(v)}{r^{2n+m+1}}\\ F_{2n}(r,v) &= r\sum_{m=0}^{\infty}\frac{F_{2n}^{m}(v)}{r^{2n+m}}\\ G_{2n}(r,v) &= -\delta_{n,1}\frac{M(v)}{r}+r\sum_{m=0}^{\infty}\frac{G_{2n}^{m}(v)}{r^{2n+m}}\end{aligned} \tag{3.25}$$

2. The functions $\Phi_{2n+1}^{m}(v)$, $F_{2n}^{m}(v)$ and $G_{2n}^{m}(v)$ are each functionals of $\psi(v)$ that scale like λ^m λ^m and λ^{m-1} under the the scaling $v\to\lambda v$.

3. For $v>\delta t$ the $\Phi_{2n+1}^{m}(v)$ are polynomials in v of degree $\leq n+m-1$; $F_{2n}^{m}(v)$ and G_{2n}^{m} are polynomials in v of degree $\leq n+m-3$ and $n+m-4$ respectively.

It follows that, say, $\phi(r,v)$, is given by a double sum

$$\phi(r,v) = \sum_{n}\Phi_{2n+1}(r,v) = \sum_{n,m=0}^{\infty}\frac{\Phi_{2n+1}^{m}(v)}{r^{2n+m+1}}.$$

Now sums over m and n are controlled by the effective expansion parameters $\sim \frac{v}{r}$ (for m) and $\frac{\psi^2 v}{\delta t r^2}\sim\frac{v}{\delta t\epsilon_f^2}\sim\frac{v}{M}$ (for n; recall that in the neighborhood of the horizon $r_H\sim\delta t\epsilon_f^2$).

It follows that the sum over m is well approximated by its first few terms if only $v\ll M$ (recall we are interested in the solution only for $r>M$). The sum over n may also be truncated to leading order only for $v \ll M$. As anticipated above, therefore, our naive perturbation expansion breaks over time scales v of order and larger than M.

Let us now focus on times v of order δt. Over these time scales naive perturbation theory is valid for $r \gg \epsilon_f \delta t$ (recall that this domain of validity includes the event horizon surface $r_H \sim \epsilon_f^2 \delta t$). Focusing on the region of interest, $r \geq r_H$, $\frac{\Phi^m_{2n+1}}{r_H^{2n+m+1}}$ scales like $\frac{1}{\epsilon_f^{2n+2m+1}}$. It follows that Φ^m_{2n+1}, with equal values of $n+m$ are comparable at times of order δt. For this reason we find it useful to define the resummed fields

$$\begin{aligned} \phi_{2n+1}(r,v) &= \sum_{k=0}^{n-1} \frac{\Phi^k_{2n+1-2k}(r,v)}{r^{2n+1-k}} \\ f_{2n}(r,v) &= r\delta_{n,2} F^0_2 + r\sum_{k=0}^{n-2} \frac{F^k_{2n-2k}(r,v)}{r^{2n-k}} \\ g_{2n}(r,v) &= r\delta_{n,2} G^0_2 + r\sum_{k=0}^{n-2} \frac{G^k_{2n-2k}(r,v)}{r^{2n-k}} \end{aligned} \tag{3.26}$$

ϕ_{2n-1}, unlike Φ_{2n-1}, receives contributions from only a finite number of terms at any fixed n, and so is effectively computable at low orders. According to our definitions, ϕ_m, f_m and g_m capture all contributions to our solutions of order $\frac{1}{\epsilon_f^m}$, at time scales of order δt.

We now present explicit computations of the fields ϕ_m, f_m and g_m up to 5th order. We find

$$\begin{aligned} f_2(r,v) &= -\frac{\psi(v)^2}{8r} \\ g_2(r,v) &= -\frac{M(v)}{r} \\ f_4(r,v) &= \frac{\psi(v)^4}{384r^3} - \frac{\psi(v)B(v)}{32r^3} \\ g_4(r,v) &= -\frac{\dot\psi(v)\psi(v)^3}{48r^3} - \frac{M(v)\psi(v)^2}{16r^3} + \frac{\dot\psi(v)B(v)}{16r^3} \\ \phi_3(r,v) &= \frac{B(v)}{4r^3} \\ \phi_5(r,v) &= \frac{\int_{-\infty}^v \left(48B(t) - 16\psi(t)^3\right)\, dt}{192r^4} \\ &\quad + \frac{\int_{-\infty}^v \left[\psi(t)\dot\psi(t)\left\{5\psi(t)^3 + 21B(t)\right\} + 3M(t)\left\{\psi(t)^3 - 18B(t)\right\}\right] dt}{192r^5} \end{aligned} \tag{3.27}$$

Where

$$B(v) = \int_{-\infty}^v \psi(t)\left(-M(t) + \psi(t)\dot\psi(t)\right)\, dt$$

3.6 Resummed perturbation theory at third order

As in the previous subsection, even at times of order δt (where naive perturbation theory is valid) naive perturbation theory yields a spacetime metric that is not uniformly well approximated by empty flat space over its region of validity $r \gg \delta t \epsilon_f$. The technical

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reason for this fact is very similar to that outlined in the previous section; g_0 is a constant, so is smaller at $r \sim r_H$ than one would have guessed from the naive extrapolation of (3.25) to $n = 0$. It follows that, in the previous section that, even at arbitrarily small ϵ, the resultant solution is well approximated by

$$g(r, v) \approx 1 - \frac{M(v)}{r}$$

rather than the flat space result $g(r, v) = 1$, over the full domain of the amplitude expansion. It follows that the correct (resummed) amplitude expansion should start with the Vaidya solution (3.22) rather than the empty flat space. The IR divergences of the naive expansion are a consequence of the incorrect choice of starting point for the perturbative expansion.

At $v = \delta t$ our metric, to leading order, is the Schwarzschild metric of a black hole Schwarzschild radius M with a superposed dilaton (and consequently metric) perturbation. We will now demonstrate that these pertubrations are small. As in the previous section, it is useful to define rescaled radial and time variables $x = \frac{r}{M}$ and $y = \frac{v}{M}$. In terms of the rescaled variables, the leading order metric takes the form

$$ds^2 = M^2 \left(2dxdy - dy^2 \left(1 - \frac{1}{x}\right) + x^2 d\Omega_2^2 \right) \tag{3.28}$$

while the ϕ perturbation is given to leading order by

$$\frac{\phi_3^0(\delta t)}{r^3} = \frac{\phi_3^0(\delta t)}{M^3 x^3} \sim \frac{1}{\epsilon_f^3 x^3} \tag{3.29}$$

(recall from (3.27) that

$$\phi_3^0(\delta t) = \frac{1}{4} \int_0^{\delta t} \psi(v) \left[-M(v) + \psi(v)\dot{\psi}(v) \right] dv \tag{3.30}$$

and $M(v)$ is given in (3.9)).

As a constant rescaling of the metric is an invariance of the equations of motion of the Einstein dilaton system, the factor of M^2 in (3.28) is irrelevant for dynamics. As the dilaton perturbation above is parametrically small ($\mathcal{O}(1/\epsilon_f^3)$) the subsequent evolution of the dilaton field is linear to leading order in the $\frac{1}{\epsilon_f}$ expansion.

Let $\chi(x, y)$ denote the unique solution to

$$\partial_x \left(x^2 \left(1 - \frac{1}{x}\right) \partial_x \chi \right) + 2x\partial_y \partial_x (x\chi) = 0 \tag{3.31}$$

subject to the boundary condition $\chi \sim \mathcal{O}(\frac{1}{x^3})$ at large x and the initial condition $\chi(x, 0) = \frac{1}{x^3}$. The leading order solution to the resummed perturbation theory for ϕ, for $v > \delta t$, is given by

$$\phi = \frac{\phi_3^0(\delta t)}{M^3} \chi \left(\frac{r}{M}, \frac{(v - \delta t)}{M} \right) \tag{3.32}$$

Unfortunately, the function $\chi(x, y)$ appears to be difficult to determine analytically. As in section 2 this solution may presumably be determined numerically with a little effort. We will not attempt the requisite numerical calculation here. In the rest of this subsection we will explain in an example how the general analysis of this subsection yields useful precise information about the subleading solution even in the absence of detailed knowledge of the function $\chi(x, y)$.

Consider a spherically symmetric shell, of the form discussed in this section, imploding inwards to form a black hole. On general grounds we expect some of the energy of the incident shell to make up the mass of the black hole, while the remaining energy is reflected back out in the form of an outgoing wave that reaches $\mathcal{I}^+$. Let the fraction of the mass that is reflected out to $\mathcal{I}^+$ be denoted by f.[20] f is one of the most interesting and easily measured observables that characterize black hole formation.

At leading order in the expansion in $\frac{1}{\epsilon_f}$ our spacetime metric takes the Vaidya form with no outgoing wave, and so $f = 0$. This prediction is corrected at first subleading order, as we now explain. It follows on general grounds that, at late times

$$\chi(x, y) \approx \frac{\zeta(y - 2x)}{x}$$

for some function $\zeta(v)$. Note that ζ, like the function χ, is universal (i.e. independent of the initial condition $\psi(v)$). It follows that at late times (and to leading order)

$$\phi = M \frac{\phi_3^0(\delta t)}{M^3} \frac{\zeta\left(\frac{v-2r}{M}\right)}{r}. \tag{3.33}$$

It then follows from (3.10) (but now applied to an outgoing rather than an ingoing wave) that the energy[21] carried by this pulse is

$$\left(M \frac{\phi_3^0(\delta t)}{M^3}\right)^2 \times \frac{1}{2} \int dt \left(\partial_t \zeta\left(\frac{t}{M}\right)\right)^2 = M \times \left(\frac{\phi_3^0(\delta t)}{M^3}\right)^2 \times \frac{1}{2} \int_{-\infty}^{\infty} dy \left(\dot{\zeta}(y)\right)^2 \tag{3.34}$$

It follows that

$$\begin{aligned} f &= A \left(\frac{\phi_3^0(\delta t)}{M^3}\right)^2 \\ A &= \frac{1}{2} \int_{-\infty}^{\infty} \dot{\zeta}^2 \end{aligned} \tag{3.35}$$

(3.35) analytically determines the dependence of f on the shape of the incident wave packet, $\psi(v)$ (recall that $\phi_3^0(\delta t)$ and M are determined in terms of $\psi(v)$ by (3.30) and (3.10)). Detailed knowledge of function $\chi(x, y)$ is required only to determine the precise value of universal dimensionless number A.

4 Spherically symmetric collapse in global AdS

We now turn to the study of black hole formation induced by an ingoing spherically symmetric dilaton pulse in an asymptotically AdS_{d+1} space in global coordinates. As in section 2

[20] In fancy parlance $f = \frac{A-B}{A}$ where A is the ADM mass of the spacetime and B is the late time Bondi mass.

[21] We have chosen our units of energy so that a black hole with horizon radius r_H has energy M.

our bulk dynamics is described by the Einstein Lagrangian with a negative cosmological constant and a minimally coupled dilaton. However as in section 3 we study solutions that preserve an SO(d) invariance; this SO(d) may be thought of as the group of rotations of the boundary S^{d-1}. As in both sections 2 and 3 our symmetry requirement determines our metric up to three unknown functions of the two variables; the time coordinate v and the radial coordinate r. Our solutions are completely determined by the boundary value, $\phi_0(v)$ of the dilaton field. As in section 2 we assume that $\phi_0(v)$ is everywhere bounded by ϵ and vanishes outside the interval $(0, \delta t)$. Through out this section we will focus on the regime $\delta t \ll R$ (where R is the radius of the boundary sphere) and $\epsilon \ll 1$. The complementary regime $\delta t \gg R$ and arbitrary ϵ is under independent current investigation [79].

The collapse process studied in this section depends crucially on two independent dynamical parameters; $x = \frac{\delta t}{R}$ together with ϵ of previous subsections. We study the evolution of our systems in a limit in which x and ϵ are both small. The problem of asymptotically AdS spherically symmetric collapse is dynamically richer than the collapse scenarios studied in sections 2 and 3, and indeed reduces to those two special cases in appropriate limits.

4.1 Set up and equations

The equations of motion for our system are given by (2.4). The form of our metric and dilaton is a slight modification of (2.6)

$$\begin{aligned} ds^2 &= 2drdv - g(r,v)dv^2 + f^2(r,v)d\Omega_{d-1}^2 \\ \phi &= \phi(r,v). \end{aligned} \tag{4.1}$$

where $d\Omega_{d-1}^2$ represents the metric of a unit $d-1$ sphere. Our fields are subject to the pure global AdS initial conditions

$$\begin{aligned} g(r,v) &= r^2 + \frac{1}{R^2}, && (v<0) \\ f(r,v) &= rR, && (v<0) \\ \phi(r,v) &= 0, && (v<0) \end{aligned} \tag{4.2}$$

and the large r boundary conditions

$$\begin{aligned} g(r,v) &= r^2\left(1+\mathcal{O}\left(\frac{1}{r^2}\right)\right) \\ f(r,v) &= r\left(R+\mathcal{O}\left(\frac{1}{r^2}\right)\right) \\ \phi(r,v) &= \phi_0(v)+\mathcal{O}\left(\frac{1}{r}\right) \end{aligned} \tag{4.3}$$

Equations (2.4), (4.1), (4.2) and (4.3) together constitute a completely well defined dynamical system. Given a particular forcing function $\phi_0(v)$, these equations and boundary conditions uniquely determine the functions $\phi(r,v)$, $g(r,v)$ and $f(r,v)$.

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The structure of the equations of motion of our system was described in subsection 2.2. In particular, we may choose the dilaton equation of motion, together with the two constraint equations, as our independent equations of motion; this set is supplemented by the energy conservation relation. With our choice of gauge and notation, the dilaton equation of motion and constraint equations take the explicit form

$$\begin{aligned}\partial_r\left(f^{d-1}g\partial_r\phi\right)+\partial_v\left(f^{d-1}\partial_r\phi\right)+\partial_r\left(f^{d-1}g\partial_v\phi\right)&=0\\(\partial_r\phi)^2+\frac{2(d-1)\partial_r^2 f}{f}&=0\\\partial_r\left(f^{d-2}g\partial_r f+2f^{d-2}\partial_v f\right)-d\,f^{d-1}-(d-2)f^{d-3}&=0\end{aligned}\tag{4.4}$$

As in section 2, the initial data needed to specify a solution to these equations is given by the value of $\phi(r)$ on a given time slice, supplemented by the initial value of the mass, and boundary conditions at infinity. In order to obtain an explicit form for the energy conservation equation we specialize to $d=3$ and explicitly 'solve' our system at large r a la Graham and Fefferman. We find

$$\begin{aligned}f(r,v)&=Rr\left(1-\frac{\dot{\phi}_0^2}{8r^2}+\mathcal{O}\left(\frac{1}{r^4}\right)\right)\\g(r,v)&=r^2\left(\frac{1}{R^2}+\frac{1-\frac{3\dot{\phi}_0^2}{4}}{r^2}-\frac{M(v)}{r^3}+\mathcal{O}\left(\frac{1}{r^4}\right)\right)\\\phi(r,v)&=\phi_0(v)+\frac{\dot{\phi}_0}{r}+\frac{L(v)}{r^3}+\mathcal{O}\left(\frac{1}{r^4}\right)\end{aligned}\tag{4.5}$$

The energy conservation equation constrains the (otherwise arbitrary) functions $M(v)$ and $L(v)$ to obey[22]

$$\dot{M}=-\frac{\dot{\phi}_0}{8}\left(12L(v)+4\frac{\dot{\phi}_0}{R^2}-3\left(\dot{\phi}_0\right)^3+4\dddot{\phi}_0(v)\right)\tag{4.7}$$

4.2 Regular small amplitude expansion

As in section 3 there are two legitimate amplitude expansions of spacetime we wish to determine. In this subsection we discuss the expansion analogous to the expansion of subsection 3.2. That is we expand all our fields as in (2.31) (where the functions f_n, g_n and ϕ_n are all defined to be of homogeneity n in the boundary field ϕ_0) and demand that all functions are everywhere regular. The requirement of regularity, together with our

[22] Note that the stress tensor and Lagrangian $\mathcal{L}$ of our system are given by

$$\begin{aligned}T_v^v&=M(v)\\T_\theta^\theta&=T_\phi^\phi=-\frac{M(v)}{2}\\\mathcal{L}&=-3L(v)-\frac{\dot{\phi}_0}{R^2}+\frac{3}{4}\left(\dot{\phi}_0\right)^3-\dddot{\phi}_0(v)\end{aligned}\tag{4.6}$$

It follows that (4.7) may be rewritten as $\dot{M}=\frac{\dot{\phi}_0\mathcal{L}}{2}$.

boundary and initial conditions, uniquely specifies all functions in (2.31). Explicitly, to second order we find

$$\begin{aligned}
\phi_1(r,v) &= \sum_{m=0}^{\infty}(-1)^m\Bigg[\phi_0(v-m\pi R)+\frac{\dot\phi_0(v-m\pi R)}{r}+\phi_0(v-Rm\pi-2R\tan^{-1}(rR)) \\
&\qquad\qquad -\frac{\dot\phi_0(v-m\pi R-2R\tan^{-1}(rR))}{r}\Bigg] \\
f_2(r,v) &= \frac{R}{4}\left(r\int_r^\infty \rho K(\rho,v)\,d\rho-\int_r^\infty \rho^2 K(\rho,v)\,d\rho\right) \\
g_2(r,v) &= -\frac{1}{4r}\Bigg[\frac{2r}{R^2}\int_r^\infty \rho K(\rho,v)\,d\rho+2r^2\int_r^\infty \rho^2 K(\rho,v)\,d\rho \\
&\qquad +\int_0^r \rho^2\left(\frac{1}{R^2}+\rho^2\right)K(\rho,v)\,d\rho\Bigg]-\frac{2}{R}\partial_v f_2(r,v) \\
&\text{where} \\
K(\rho,v) &= (\partial_r\phi_1(r,v))^2
\end{aligned} \tag{4.8}$$

The perturbation expansion in this section is valid only if $\phi(r,v)$ is everywhere small on the solution. $\phi_1(r,v)$ reaches its maximum value in the neighborhood of the origin where it is given approximately by $\phi_0+\ddot\phi_0\sim\epsilon+\frac{\epsilon}{x^2}$. Consequently the validity of the amplitude expansion sketched in this section requires both that $\epsilon\ll 1$ and that $x^2\gg\epsilon$.

We have chosen integration constants to ensure that the solution in (4.8) is regular at $r=0$. In particular

$$g_2(0,v)=\frac{1}{2}\left(\int_0^\infty \rho^2\partial_v K(\rho,v)\,d\rho-\frac{1}{R^2}\int_0^\infty \rho K(\rho,v)\,d\rho\right).$$

As in subsection 3.2, this choice automatically implies the energy conservation equation. In particular, expanding $g_2(r,v)$ at large r we find

$$-M(v)=-\frac{1}{4}\left(\int_0^\infty \rho^2\left(\frac{1}{R^2}+\rho^2\right)K(\rho,v)\,d\rho-\dot\phi_0(v)^2-2\dot\phi_0(v)\dddot\phi_0(v)\right) \tag{4.9}$$

(this equation is valid only for $v<\pi R$; it turns out that $M(v)$ is constant for $v>\delta t$) in agreement with the energy conservation equation.

Finally, let us focus on the coordinate range $rR,\frac{v}{R}\ll 1$ and also require that x is small so that the time scale in ϕ_0 is also smaller than the AdS radius. In this parameter and coordinate range (4.8) should reduce to a solution of the flat space propagation equation (3.11); this is easily verified to be the case. In the given variable and parameter regime, all terms with (4.8) with $m\neq 0$ vanish; $\tan^{-1}(Rr)\approx rR$ and the first and the third terms in (4.8) are negligible compared to the second and fourth as x is small. Putting all this together, (4.8) reduces to (3.11) under the identification $\psi(v)=R^2\dot\phi_0(v)$, once we also identify the coordinate r of section 3 with R^2r in this section. Notice that this replacement implies that $\epsilon_f=\frac{\epsilon}{x^2}$ (where ϵ_f was the perturbative expansion of section 3). This identification of parameters is consistent with the fact that the expansion of this section breaks down when $\frac{\epsilon}{x^2}$ becomes large, while the expansion of subsection 3.2 breaks down at large ϵ_f.

4.3 Spacetime and event horizon for black hole formation

In the rest of this section we will describe the process of black hole formation via collapse in an amplitude expansion. As in earlier sections, the spacetime that describes this collapse process will turn out to be given, to leading order, by the Vaidya form

$$
\begin{aligned}
ds^2 &= 2drdv - \left(\frac{1}{R^2} + r^2 - \frac{M(v)}{r}\right) dv^2 + R^2 r^2 d\Omega_2^2 \\
\phi(r,v) &= \phi_0(v) + \frac{\dot{\phi}_0}{r}
\end{aligned}
\tag{4.10}
$$

where $M(v)$ is approximated by $C_2(v)$, the order ϵ^2 piece of (4.7)

$$
C_2(v) = -\frac{1}{2}\int_{-\infty}^{v} dt \dot{\phi}_0(t)\left(\dddot{\phi}_0(t) + \frac{\dot{\phi}_0(t)}{R^2}\right) \tag{4.11}
$$

In this subsection we will compute the event horizon of the spacetime (4.10) in a perturbation expansion in a small parameter, whose nature we describe below. The horizon is determined by the differential equation

$$
2\frac{dr_H}{dv} = \frac{1}{R^2} + r_H^2 - \frac{M(v)}{r_H} \tag{4.12}
$$

where $M(v)$ reduces to a constant M for $t > \delta t$. At late times the event horizon surface must reduce to the largest real solution of the equation

$$
\frac{1}{R^2} + (r_H^0)^2 - \frac{M}{r_H^0} = 0.
$$

It then follows from (4.12) that

$$
\begin{aligned}
r_H(v) &= r_H^0, && (v > \delta t) \\
r_H(v) &= r_H^0 x\left(\frac{v}{\delta t}\right), && (0 < v < \delta t) \\
\tan^{-1}\left(r_H(v)\right) &= \tan^{-1}\left(r_H^0 x(0)\right) + v && (v < 0), \quad \tan^{-1}(r_H(v)) > 0
\end{aligned}
\tag{4.13}
$$

As in previous subsections, the function $x(t)$ is easily generated in a perturbation expansion

$$
x(t) = 1 + \left(\frac{M\delta t}{(r_H^0)^2}\right) x_1(t) + \left(\frac{M\delta t}{(r_H^0)^2}\right)^2 x_2(t) + \ldots \tag{4.14}
$$

The small parameter for this expansion is $\frac{M\delta t}{(r_H^0)^2}$. This parameter varies from approximately $\epsilon^{\frac{2}{3}}$ when $x \ll \epsilon^{\frac{2}{3}}$ to $\frac{x^4}{\epsilon^2}$ when $x \gg \epsilon^{\frac{2}{3}}$ and is always small provided $x \ll \sqrt{\epsilon}$ and $\epsilon \ll 1$. These conditions will always be met in our amplitude constructions below. Note that the event horizon of our solution is created (at $r = 0$) at the time $v = -\tan^{-1}(r_H^0)+$ subleading.

Explicitly working out the perturbation series we find

$$
\begin{aligned}
x_1(t) &= -\int_t^1 dt \frac{1 - \frac{M(y\delta t)}{M}}{2} \\
x_2(t) &= -\int_t^1 dy \left(\frac{2(r_H^0)^3}{M} + \frac{M(y\delta t)}{M}\right)
\end{aligned}
\tag{4.15}
$$

4.4 Amplitude expansion for black hole formation

The amplitude expansion of the previous subsection breaks down for $x^2 \ll \epsilon$. As in section 3, we have a new amplitude expansion in this regime. As in section 3, the starting point for this expansion is the Vaidya metric and dilaton field (4.10). As in sections 2 and 3, the perturbation expansion based on (4.10) is technically difficult to implement at late times. However as in earlier sections, at early times — i.e. times of order δt — the perturbative expansion is well approximated by the naive expansion based on the solution (4.10) with $M(v)$ set equal to zero. Following the terminology of previous sections we refer to this simplified expansion as the naive expansion. In the rest of this subsection we will elaborate on the analytic structure of the naive perturbative expansion.

In order to build the naive expansion, we expand the fields $f(r,v)$, $g(r,v)$ and $\phi(r,v)$ in the form (2.31). It is not too difficult to inductively demonstrate that

1. The functions ϕ_{2n+1}, g_{2n} and f_{2n} have the following analytic structure in the variable r

$$\begin{aligned}
\phi_{2n+1}(r,v) &= \sum_{m=0}^{\infty} \frac{1}{R^{2m}} \sum_{k=0}^{2n+m-2} \frac{\phi_{2n+1}^{k,m}(v)}{r^{2n+1-k+m}} && (n \geq 1) \\
f_{2n}(r,v) &= rR \sum_{m=0}^{\infty} \frac{1}{R^{2m}} \sum_{k=0}^{2n-4} \frac{f_{2n}^{k,m}(v)}{r^{2n-k+m}} && (n \geq 2) \qquad (4.16) \\
g_{2n}(r,v) &= -\frac{C_{2n}(v)}{r} + r \sum_{m=0}^{\infty} \frac{1}{R^{2m}} \sum_{k=0}^{2n-3} \frac{g_{2n}^{k,m}(v)}{r^{2n-k+m}} && (n \geq 2)
\end{aligned}$$

2. The functions $\phi_{2n+1}^{k,m}(v)$, $f_{2n}^{k,m}(v)$ and $g_{2n}^{k,m}(v)$ are functionals of $\phi_0(v)$ that scale like $\lambda^{-2n-1+m+k}$, $\lambda^{-2n+m+k}$ and $\lambda^{-2n+m+k-1}$ respectively under the scaling $v \to \lambda v$.

3. For $v > \delta t$ we have some additional simplifications in structure. At these times $f_4(r,v) = 0$ and $g_4(r,v) = -\frac{C_4(v)}{r}$. Further, the sums over k in the second and third of the equations above run from 0 to $2n-6+m$ and $2n-5+m$ respectively. Finally, functions $\phi_{2n+1}^{k,m}(v)$ are all polynomials in v of a degree that grows with n. In particular the degree of $\phi_{2n+1}^{k,m}$ is at most $n-1+k+m$; the degree of $f_{2n}^{k,m}$ is at most $n-3+k+m$ and the degree of $g_{2n}^{k,m}$ is at most $n-4+k+m$.

As we have explained above,

$$\phi(r,v) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{R^{2m}} \sum_{k=0}^{2n-2+m} \frac{\phi_{2n+1}^{k,m}(v)}{r^{2n+1-k+m}}.$$

We will now discuss the relative orders of magnitude of different terms in this summation. Abstractly, at times that are larger than or of order δt, the effective weighting factor for the sum over n, m, k respectively are approximately given by $\frac{\epsilon^2 v}{r^2(\delta t)^3}$, $\frac{v}{R^2 r}$ and vr respectively. We will try to understand the implications of these estimates in more detail.

Let us first suppose that $x \ll \epsilon^{\frac{2}{3}}$. In this case the black hole that is formed has a horizon radius of order $\frac{\epsilon^{\frac{2}{3}}}{\delta t} \gg \frac{1}{R}$ (this estimate is corrected in a power series in $\frac{x^2}{\epsilon^{\frac{4}{3}}}$). Consequently, the resultant black hole is large compared to the AdS radius. At $m = 0$, this regime, the summation over k and n simply reproduce the solution of section 2. As in section 2 these summations are dominated by the smallest values of k and n for $vr_H \sim vT \ll 1$, in the neighborhood of the horizon. As in section 2 the sum over k is dominated by the largest value of k at large enough r. The new element here is the sum over m; this summation is dominated by small m when $vT \ll \frac{\epsilon^{\frac{4}{3}}}{x^2}$. When $x \ll \epsilon^{\frac{2}{3}}$, this condition automatically follows whenever $vT \ll 1$. Consequently, naive perturbation theory is always good for times small compared to the inverse black hole temperature, in this regime.

We emphasize that naive perturbation theory is always good at times of order δt. Over such time scales (and for $r \sim r_H$) we note that the sum over n and k are weighted by $\epsilon^{\frac{2}{3}}$ (this is as in section 2) while the sum over m is weighted by $\epsilon^{\frac{2}{3}}(\frac{x}{\epsilon^{\frac{2}{3}}})^2$. Note that the weighting factor for the sum over m is smaller than the weighting factor for the sum over, for instance n, provided $x \ll \epsilon^{\frac{2}{3}}$. It follows that our naive perturbation theory represents a weak departure from the black brane formation solution of section 2 when $x \ll \epsilon^{\frac{2}{3}}$.

Now let us turn to the the parameter regime $x \gg \epsilon^{\frac{2}{3}}$. In this regime $r_H R \sim \frac{\epsilon^2}{x^3}$, so that black holes that are formed in the collapse process are always small in units of the AdS radius. At times that are larger or of order δt, the sum over m and n are dominated by their smallest values provided $\frac{v}{R} \ll \frac{\epsilon^2}{x^3}$. Making the replacement $\epsilon = x^2 \epsilon^f$, this condition reduces to $v \ll \epsilon_f^2 \delta t$ which was exactly the condition for applicability of naive perturbation theory in flat space in section 3. The new element here is the sum over k. k is zero in section 3, and the sum over k here is dominated by $k = 0$ near $r = r_H$ for $\frac{v}{R} \ll \frac{x^3}{\epsilon^2}$, a condition that is automatically implied by $\frac{v}{R} \ll \frac{\epsilon^2}{x^3}$. Note, however, that, as in the previous paragraph, the sum over k is always dominated by the largest value of k at sufficiently large r. This reflects the fact that AdS space is never well approximated by a flat bubble at large r. Finally, specializing to v of order δt and $r \sim r_H$, the sum over n and m are each weighted by $\frac{x^4}{\epsilon^2} \sim \frac{1}{\epsilon_f^2}$ while the sum over k is weighted by $\epsilon \frac{\epsilon}{x^2}$. In particular naive perturbation theory is good at times of order δt provided $x \ll \sqrt{\epsilon}$.

Let us summarize in broad qualitative terms. Naive perturbation theory is a good expansion to the true solution when $vT \ll 1$ for $\frac{v}{R} \ll \frac{\epsilon^2}{x^3}$. In particular, this condition is always obeyed for times of order δt when $x \ll \sqrt{\epsilon}$.

4.5 Explicit results for naive perturbation theory

As we have explained above, the functions ϕ, f and g may be expanded in an expansion in ϵ as

$$\begin{aligned}
\phi(r,v) &= \epsilon\phi_1(r,v) + \epsilon^3\phi_3(r,v) + \mathcal{O}(\epsilon)^5 \\
f(r,v) &= rR\left(1 + \epsilon^2 f_2(r,v) + \epsilon^4 f_4(r,v) + \mathcal{O}(\epsilon)^6\right) \\
g(r,v) &= r^2 + \frac{1}{R^2} + \epsilon^2 g_2(r,v) + \epsilon^4 g_4(r,v) + \mathcal{O}(\epsilon)^6
\end{aligned} \tag{4.17}$$

Moreover the functions ϕ_{2n+1}, f_n and g_n may themselves each be expanded as a sum over two integer series (see (4.16)). The sum over k runs over a finite number of values in (4.16) and we will deal with this summation exactly below. However the sum over m runs over all integers, and is computatble only after truncation to some finite order. This truncation is justified as the sum over m is effectively weighted by a small parameter as explained in the section above. In this section we present exact expressions for the functions ϕ_1, g_2 and f_2, and expressions for ϕ_3, f_4 and g_4 to the first two orders in the expansion over the integer m (this summation is formally weighted by $\frac{1}{R}$);

The solutions are given as

$$
\begin{aligned}
\phi_1(r,v) &= \phi_0(v) + \frac{\dot\phi_0(v)}{r} \\
f_2(r,v) &= -\frac{\dot\phi_0^2}{8r^2} \\
g_2(r,v) &= -\frac{3\dot\phi_0^2}{4} - \frac{C_2(v)}{r} \\
\phi_3(r,v) &= \frac{K(v)}{r^3} \\
&\quad + \frac{1}{R^2}\left[\frac{\int_{-\infty}^{v}\left(3K(t)-\dot\phi_0(t)^3\right)dt}{12r^4} + \frac{\int_{-\infty}^{v}dt_1\int_{-\infty}^{t_1}dt_2\left(3K(t_2)-\dot\phi_0(t_2)^3\right)}{12r^3}\right] \\
&\quad + \mathcal{O}\left(\frac{1}{R}\right)^4 \\
f_4(r,v) &= \left(\frac{\dot\phi_0^4}{384r^4} - \frac{A_3(v)}{32r^4}\right) + \frac{1}{R^2}\left(\frac{A_1(v)}{96r^4} + \frac{A_2(v)}{120r^5}\right) + \mathcal{O}\left(\frac{1}{R}\right)^4 \\
g_4(r,v) &= -\frac{C_4(v)}{r} + \frac{3A_3(v)-\dot\phi_0^4}{24r^2} + \frac{1}{48r^3}\left(3\dot A_3(v) - 4\dot\phi_0^3\ddot\phi_0\right) \\
&\quad - \frac{1}{R^2}\left[\frac{A_1(v)}{24r^2} + \frac{\dot A_1(v)}{48r^3} + \frac{15A_3(v)+4A_2(v)-\dot\phi_0^4}{240r^4}\right] + \mathcal{O}\left(\frac{1}{R}\right)^4
\end{aligned}
\tag{4.18}
$$

Where

$$
\begin{aligned}
K(v) &= \int_{-\infty}^{v} dt\ \dot\phi_0\left(-C_2(t) + \dot\phi_0\ddot\phi_0\right) \\
A_1(v) &= \dot\phi_0(v)\int_{-\infty}^{v} dt_1 \int_{-\infty}^{t_1} dt_2\left(-3K(t_2) + \dot\phi_0^3(t_2)\right) \\
A_2(v) &= \dot\phi_0(v)\int_{-\infty}^{v} dt\left(-3K(t) + \dot\phi_0^3(t)\right) \\
A_3(v) &= \dot\phi_0 K(v)
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
C_2(v) &= -\frac{1}{2}\int_{-\infty}^{v} dt\ \dot\phi_0(t)\left(\frac{\dot\phi_0(t)}{R^2} + \dddot\phi_0(t)\right) \\
C_4(v) &= -\frac{3}{8}\int_{-\infty}^{v} dt\ \dot\phi_0(t)\left(K(t) - \dot\phi_0(t)^3\right) \\
&\quad -\frac{1}{8R^2}\int_{-\infty}^{v} dt_1\ \dot\phi_0(t_1)\int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3\left(3K(t_3) - \dot\phi_0(t_3)^3\right) + \mathcal{O}\left(\frac{1}{R}\right)^4
\end{aligned}
\tag{4.20}
$$

4.6 The solution at late times

As in previous sections, our solution is normalizable (unforced) for $v > \delta t$. Naive perturbation theory reliably establishes the initial conditions for this unforced evolution at $v = \delta t$. To leading order, this evolution is given by global AdS black hole metric with $M = C_2(\delta t)$ (see (4.11)), perturbed by $\phi(\delta t) = \frac{K(\delta t)}{r^3}$ see (4.18). As in the previous two subsections, the qualitatively important point is that this represents a small perturbation about the black hole background. Moreover, it follows on general grounds that perturbations in a black hole background in AdS space never grow unboundedly (in fact they decay) with time. Consequently, we may reliably conclude that our spacetime takes the Vaidya form (4.10) at all times to leading order in the amplitude expansion.

In order to determine an explicit expression for the subsequent dilaton evolution, one needs to solve for the linear, minimally coupled, evolution of a $\frac{1}{r^3}$ initial condition in the background of global AdS with a Schwarzschild black hole of arbitrary mass. As in the previous two sections, the linear differential equation one needs to solve appears to be analytically intractable, but could easily be solved numerically. We will not, however, attempt this evaluation in this paper.

5 Discussion

In this paper we have used the AdS/CFT correspondence to determine the response of a conformal field theory, initially in its vacuum, to a low amplitude perturbation by a source coupled to a marginal operator. When the CFT in question lives on $R^{d-1,1}$ it responds to the perturbation by the source by thermalizing into a plasma type phase. On the other hand, when the CFT in question lives on a sphere it either thermalizes into a plasma type phase or settles down into a glueball type phase depending on the details of the perturbation procedure. In this paper we have demonstrated that, to leading order in the amplitude expansion, the dual description of the thermalization into a plasma type phase is a spacetime of the Vaidya form. In odd boundary field theory dimensions the Vaidya metric reduces exactly to the uniform black brane metric in the causal future $v > \delta t$ at the boundary. As was discussed in detail in section 1.1, for many purposes our system behaves as if it has thermalized *instantaneously*.

In this paper we have only studied solutions with a high degree of symmetry; for instance, solutions that maintain spatial translational invariance. The solutions of this paper may prove to be a useful starting point in describing the response of the field theory to a forcing function that breaks this symmetry, provided the scale of spatial variation of the forcing function is large compared to the inverse temperature of the black brane that is set up in our solutions. Consider, for example, the Einstein dilaton system studied in section 2 perturbed by the non normalizable part of a small amplitude dilaton field that takes the form $\phi_0(v, \vec{x})$. Let us further assume that the length scales for spatial variation $L(\vec{x})$ in ϕ_0 are all large compared to $\frac{\delta t(\vec{x})}{\epsilon^{\frac{2}{3}}}$ (the inverse of the temperature of the black brane that is eventually formed). As $\epsilon \ll 1$ this implies, in particular, that $L(\vec{x}) \gg \delta t(\vec{x})$. We expect the resultant thermalization process to be described by a dual metric of the form

$$ds^2 = 2drdv - \left(r^2 - \frac{M(v,\vec{x})}{r^{d-3}}\right) dv^2 + r^2 dx_i^2. \tag{5.1}$$

where

$$\begin{aligned} M(v,\vec{x}) &= C_2(v,\vec{x}) + \mathcal{O}(\epsilon^4) \\ C_2(v,\vec{x}) &= -\frac{1}{2}\int_{-\infty}^{v} dt\dot{\phi}_0(t,\vec{x})\dddot{\phi}_0(t,\vec{x}) \\ C_2(\vec{x}) &= \frac{1}{2}\int_{-\infty}^{\infty} dt\ddot{\phi}_0(t,\vec{x})\ddot{\phi}_0(t,\vec{x}) \end{aligned} \tag{5.2}$$

i.e. to be approximated tubewise by the solutions described in this paper. The metric (5.1) will then be corrected in a power series expansion in two variables; ϵ (as in this paper) and a spatial derivative expansion weighted by $\frac{\delta t}{L\epsilon^{\frac{2}{3}}}$. The last expansion should reduce to the fluid dynamical expansion at late times. Indeed, at $t = \delta t$, the metric described in (5.1) is dual to a locally thermalized conformal fluid, everywhere at rest, but with a space varying energy density $C_2(\vec{x})$. The evolution of this fluid after $v = \delta t$ will simply be governed by the Navier Stokes equations of fluid dynamics; the metric dual to the corresponding flow was determined in [1–9, 9–19]. Provided we can solve the relevant fluid dynamical equations, we have a complete description of the evolution of our spacetime for all v.

The gravitational solutions presented in this paper appear to be qualitatively different, in several ways, for odd and even d (see appendix B.1). This suggests that the equilibration process at strong coupling is qualitatively different in odd and even dimensional field theories. At leading order in amplitudes, equilibration takes place faster in in field theories with odd spacetime dimensions as compared to their even dimensional counterpart. It would be interesting to find a direct field theory explanation of this fact.

In this paper we have investigated the response of an AdS space to a marginal, non normalizable deformation of small amplitude. It would be natural to extend our work to determine the response of the same space to a relevant or irrelevant non normalizable marginal deformation of small amplitude. More ambitiously, one could also hope to explore the response of the system to large amplitude deformations, perhaps using a combination of analytic and numerical techniques (see [53]). We leave these issues to the future.

In this paper we have constructed several solutions to bulk equations in a perturbative expansion. It is natural to wonder whether the dynamical processes we have constructed in this paper are stable to small fluctuations, when embedded into familiar examples of the AdS/CFT correspondence. We will not address this question in detail in this paper; in these paragraphs we simply address the question of when the end point of the collapse processes, studied in this paper, are stable.

In this paper we have described time evolutions that end up in big black holes, small black holes (big and small compared to the AdS radius) and a thermal gas in AdS. To the best of our knowledge, large uncharged AdS black holes are stable solutions in every familiar example of the AdS/CFT correspondence. The AdS thermal gas has a potential instability, the Jeans instability, which is triggered at energies at or larger than a critical density of

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order unity [78], in the units of our paper. However the collapse situations described in this paper, that end up in a thermal gas, do so at energies of order $\frac{\epsilon^2}{x^d} \ll \epsilon^{\frac{d-2}{d-1}} \ll 1$. We conclude that the thermal gases produced as the end point of collapse, in our paper, are also stable.

Small black holes in $AdS_{d+1} \times X$, on the other hand, are usually unstable to a Gregory Laflamme type clumping on the internal manifold X [80]. Consequently, the evolutions leading to small AdS black holes, constructed in this paper, are necessarily unstable when embedded into familiar examples of the AdS/CFT correspondence.[23] Note, however, that the time scale associated with this Gregory Laflamme instability is $R^2 r_H$ where r_H is the Schwarszchild radius of the small black hole. Assuming $Rr_H \ll 1$ (so that the black hole that is formed is really small), $\frac{\delta t}{r_H R^2} \sim \left(\frac{x^{d-1}}{\epsilon}\right)^{\frac{2}{d-2}}$. It follows that, in the limit $x^{d-1} \ll \epsilon$, studied in this paper, the black hole formation processes discussed in this paper occur over a time scale much smaller than that of Gregory Laflamme instability. In other words the thermalization to small black holes (in the perturbative regime described in this paper) is described by a two stage process. In the first stage the solution is well described by the Vaidya metric (plus corrections) described in this paper. The second stage describes the evolution of an almost completely formed small AdS black hole perturbed by the Gregory Laflamme instability. This black hole will then undergo the Gregory Laflamme type transition in the usual manner. In other words the perturbative solutions presented in this paper correctly describe the process of small black hole formation even when embedded in AdS/CFT type situations in which this black hole is unstable.

As we have explained above, a CFT on the sphere can respond to a forcing function by settling down into either of its two available phases. It appears that the space of possible forcings is divided into two regions, by a critical surface of unit codimension. On either side of this critical surface, the forcing drives the system into different phases. This critical surface occurs at $x^{d-1} \sim \epsilon^2$. Ignoring the issue of the Gregory Laflamme instability for a moment, the local (small r) form of this gravitational solution precisely on this critical surface is simply Choptuik's critical solution in $R^{d-1,1}$, which is known to display surprisingly robust universal behavior characterized by a universal, nakedly singular solution and universal critical exponents. Note, however, that the Gregory Laflamme instability generically cannot be ignored in the neighbourhood of the critical surface. In the neighbourhood of this transition black holes that are formed near criticality have arbitrarily small Schwarzscild radius, and so trigger the Gregory Laflamme instability over very short time scales. Consequently, in order to access the universal $R^{d-1,1}$ Choptuik behavior in field theory one would have to tune initial data with exponential accuracy.

It is of course true on general grounds that the solutions described in this paper fall into two classes distinguished by an order parameter (the presence of a horizon at late times). An important question about the transition between these two behaviours is whether it is continuous or discontinuous over time scales small compared to $\frac{1}{R}$. In situations in which Gregory Laflamme instability occurs there is a very special submanifold of this transition manifold; the submanifold on which Gregory Laflamme instabilities are precisely tuned

[23] We thank O. Aharony and B. Kol for discussions on this point.

away. On this submanifold we know that the relevant gravitational solutions are those of Choptuik collapse in $R^{d-1,1}$, and so are continuous (second order) and singular. We do not konw if this singular second order behaviour persists away from this special point. The investigation of these issues, as well as the study of the smoothing out of singularities on this manifold by finite N fluctuations, is potentially interesting area of future research.

Acknowledgments

We would like especially to thank O. Aharony, B. Kol and S. Trivedi for several interesting discussions. We would also like to thank J.Bhattacharya, S. Dutta, R. Loganayagam, P. Joshi, G. Mandal, S. Raju and S. Wadia and all the participants of 'String Theory and Fundamental Physics' at the Kanha Tiger sanctuary for useful discussions. We also thank O. Aharony, J. Bhattacharya, S. Das, G. Horowitz, V. Hubeny, B. Kol, R. Loganayagam, G. Mandal, T. Wiseman, M. Rangamani for detailed comments on a preliminary version of this manuscript. We thank H. Antia and M. Mahato for helping in the numerical solution of the differential equation in section 2. We would also like to acknowledge useful discussion with the students in the TIFR theory room. The work of S.M. was supported in part by Swarnajayanti Fellowship. We would all also like to acknowledge our debt to the people of India for their generous and steady support to research in the basic sciences.

A Translationally invariant graviton collapse

In sections 2 and 4 above we have studied the collapse triggered by a minimally coupled scalar wave in an asymptotically AdS background. Our study was, in large part, motivated by potential applications to CFT dynamics via the AdS/CFT correspondence. From this point of view the starting point of our analyses in e.g. section 2 has a drawback as not every bulk system that arises in the study of the AdS/CFT correspondence, admits a consistent truncation to the theory of gravity coupled to a minimally coupled massless scalar field.

On the other hand, every two derivative theory of gravity that admits AdS space as a solution admits a consistent truncation to Einstein gravity with a negative cosmological constant. Consequently, any results that may be derived using the graviton instead of dilaton waves, applies universally to all examples of the AdS/CFT correspondence with two derivative gravity duals. In this section we study a situation very analogous to the set up of section 2, with, however, a transverse graviton playing the place of the dilaton field of section 2. All the calculations of this section apply universally to any CFT that admits a two derivative gravitational dual.

While the equations that describe the propagation of gravity waves are more complicated in detail than those that describe the propagation of a massless minimally coupled scalar field, it turns out that the final results of the calculations presented in this subsection are extremely similar to those of section 2. We take this to suggest that all the qualitative results of sections 2 and 4 would continue to qualitatively apply to the most general approximately translationally invariant gravitational perturbations of Poincare Patch AdS space or approximately spherically symmetric gravitational perturbation of global AdS space. If this guess is correct, it suggests that the qualitative lessons learnt in this paper have a wide degree of applicability.

In this section we restrict our attention to the simplest dimension $d = 3$. It should we possible, with some additional effort, to extend our results at least to all odd d, and also to work out the corresponding results for even d. We leave this extension to future work.

The set up of this appendix is very closely analogous to that employed by Yaffe and Chesler in [53]. The main differences are as follows. Yaffe and Chesler worked in $d = 4$; they numerically studied the effect of a specific large amplitude non normalizable deformation on the gravitational bulk. We work in $d = 3$, and analytically study the the effect of the arbitrary small amplitude deformation on the gravitational bulk.

A.1 The set up and summary of results

In this section we study solutions to pure Einstein gravity with a negative cosmological constant. We study solutions that preserve an $R^2 \times Z_2 \times Z_2$ symmetry. Here R^2 denotes the symmetry of translations in spatial field theory directions, while the two Z_2s respectively denote the spatial parity flip and the discrete exchange symmetry between the two Cartesian spatial boundary coordinates x and y.

As in section 2, our symmetry requirements determine our metric up to three unknown functions of v and r. With the same choice of gauge as in section 2, our metric takes the form

$$ds^2 = -2\ dv\ dr + g(r,v)\ dv^2 + f^2(r,v)(dx^2 + dy^2) + 2r^2 h(r,v) dx\ dy \qquad \text{(A.1)}$$

The boundary conditions on all fields are given by (2.9) under the replacement $\phi(r,v) \to h(r,v)$ and $\phi_0(v) \to h_0(v)$. Here $h_0(v)$ gives the boundary conditions on the off diagonal mode, g_{xy}, of the boundary metric. $h_0(v)$ is taken to be of order ϵ. Physically, our boundary conditions set up a graviton wave, with polarization parallel to the spatial directions of the brane.

As in section 2, in order to solve Einstein's equations with the symmetries above, it turns out to be sufficient to solve the three equations E_C^2, E_C^1 and E_{xy} (see (2.10)) (plus the energy conservation condition rE_{ec} at one r).

As in section 2 it is possible to solve these equations order by order in ϵ. We present our solution later in this section. To end this subsection, we list the principal qualitative results of this section. We are able to show that

- The boundary conditions described above result in black brane formation for an arbitrary (small amplitude) source function $h_0(v)$.
- Outside the event horizon of our spacetime, we find an explicit analytic form for the metric as a function of $h_0(v)$. Our metric is accurate at leading order in the ϵ expansion, and takes the Vaidya form (1.1) with a mass function

$$M(v) = -\frac{1}{2}\int_{-\infty}^{v} dt \dot{h}_0 \dddot{h}_0 \qquad \text{(A.2)}$$

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- In particular, we find that the energy density of resultant black brane is given by

$$M \approx -E_2 = \frac{1}{2}\int_{-\infty}^{\infty} dt \ddot{h}_0^2 \tag{A.3}$$

Note that $E_2 \sim \frac{\epsilon^2}{(\delta t)^3}$.

- As this leading order metric is of the same form as that in the previous subsection, the analysis of the event horizons presented above continues to apply. In particular it follows that singularities formed in the process of black brane formation are always shielded by a regular event horizon at small ϵ.

- Going beyond leading order, perturbation theory in the amplitude ϵ yields systematic corrections to this metric at higher orders in ϵ. We unravel the structure of this perturbation expansion in detail and work out this perturbation theory explicitly to fifth order at small times.

A.2 The energy conservation equation

As we have explained above, the equations of motion for our system include the energy conservation relation, in addition to the one dynamical and two constraint equations. The form of the dynamical and constraint equations is easily determined using Mathematica-6; these equations turn out to be rather lengthy and we do not present them here. In this section we content ourselves with presenting an explicit form for the energy conservation equation. As in section 2, it is possible to solve for the functions $\frac{f}{r}$, $\frac{g}{r^2}$ and h in a power series in $\frac{1}{r}$. This solution is simply the Graham Fefferman expansion. To order $\frac{1}{r^3}$ (relative to the leading result) we find

$$\begin{aligned}
f(r,v) &= r\left(1+\frac{\frac{\left[\dot{h}_0\right]^2}{8\left(1-h_0^2\right)}}{r^2}+\frac{\frac{1}{2}h_0\sigma(v)}{r^3}+\cdots\right)\\
g(r,v) &= r^2\left(1+\frac{\frac{1}{4\left(-1+h_0^2\right)^2}\left[\left(1+3h_0^2\right)\left[\dot{h}_0\right]^2-4h_0\left(-1+h_0^2\right)\partial_v^2 h_0\right]}{r^2}-\frac{M(v)}{r^3}+\cdots\right)\\
h(r,v) &= \left(h_0+\frac{\dot{h}_0}{r}+\frac{\frac{h_0\dot{h}_0^2}{4\left(-1+h_0^2\right)}}{r^2}+\frac{\sigma(v)}{r^3}+\cdots\right)
\end{aligned} \tag{A.4}$$

where the parameters M and σ are constrained by the energy conservation equation[24]

$$\dot{M} = -\frac{\dot{h}_0}{2\left(-1+h_0^2\right)^4}\Big[+3M(v)h_0\left(-1+h_0^2\right)^3 - 3\left(-1+h_0^2\right)^3\sigma - 4\left(-1+h_0^2\right)h_0\dot{h}_0\partial_v^2 h_0 + \left(-1+h_0^2\right)^2\partial_v^3 h_0 + \left(1+3h_0^2\right)\left[\dot{h}_0\right]^3\Big] \tag{A.6}$$

In the perturbative solution we list below, we will find that $\sigma \sim \mathcal{O}(\epsilon^3)$. It follows that, to order $\mathcal{O}(\epsilon^2)$, the function $M(v)$ is given by (A.2).

A.3 Structure of the amplitude expansion

As in subsection 2 we set up a naive amplitude expansion by formally replacing h_0 with ϵh_0 and then solving our equations in a power series in ϵ. We expand

$$\begin{aligned} f(r,v) &= \sum_{n=0}^{\infty} \epsilon^n f_n(r,v) \\ g(r,v) &= \sum_{n=0}^{\infty} \epsilon^n g_n(r,v) \\ h(r,v) &= \sum_{n=0}^{\infty} \epsilon^n h_n(r,v) \end{aligned} \tag{A.7}$$

with

$$f_0(r,v) = r, \quad g_0(r,v) = r^2, \quad h_0(r,v) = 0. \tag{A.8}$$

The formal structure of this expansion is identical to that described in section 2.5; in particular f_n and g_n are nonzero only for even n while h_n is nonzero only for odd n. At first order we find

$$h_1(r,v) = h_0(r,v) + \frac{\dot{h}_0(r,v)}{r} \tag{A.9}$$

which then leads to simple expressions (see below) for f_2 and g_2. In particular h_1 and f_2 vanish for $v \geq \delta t$ while $g_2 = M/r$ for $v \geq \delta t$.

Turning to higher orders in the perturbative expansion, it is possible to inductively demonstrate that for $n \geq 1$

[24] The stress tensor is given by

$$\begin{aligned} T_{tt} &= M \\ T_{xx} &= T_{yy} = -\frac{M}{2} \\ T_{xy} &= -\frac{1}{2\left(-1+h_0^2\right)^3}\Big[-3\left(-1+h_0^2\right)^3\sigma(v) - 4\left(-1+h_0^2\right)h_0\dot{h}_0\partial_v^2 h_0 + \left(-1+h_0^2\right)^2 h_0^3 + \left(1+3h_0^2\right)\left[\dot{h}_0\right]^3\Big] \end{aligned} \tag{A.5}$$

Using these relations, it may be verified that (A.6) is simply a statement of the conservation of the stress tensor.

1. The functions h_n, g_n and f_n have the following analytic structure in the variable r

$$\begin{aligned} h_{2n+1}(r,v) &= \sum_{k=2}^{2n+1} \frac{\phi_n^k(v)}{r^k} \\ f_{2n+2}(r,v) &= r\sum_{k=2}^{2n+2} \frac{f_n^k(v)}{r^k} \\ g_{2n+2}(r,v) &= r\sum_{k=1}^{n} \frac{g_n^k(v)}{r^k} \end{aligned} \tag{A.10}$$

2. The functions $h^k_{2n+1}(v)$, $f^k_{2n+2}(v)$ and $g^k_{2n+2}(v)$ are each functionals of $h_0(v)$ that scale like λ^{-k} under the scaling $v \to \lambda v$.

3. For $v > \delta t$ these functions are all polynomials in v of a degree that grows with n. For example, the degree of h^k_{2n+1} is of at most $3n - k$.

As in the section 2, this structure ensures that naive perturbation theory is good for times $v \ll M^{\frac{1}{3}}$, but fails for later times. As in section (2), the correct perturbative expansion uses the Vaidya metric (1.1) as the zero order solution.

A.4 Explicit results up to 5th order

At leading order we have

$$\begin{aligned} h_1(r,v) &= h_0(v) + \frac{\dot{h}_0}{r} \\ f_2(r,v) &= \frac{\left[\dot{h}_0\right]^2}{8r} \\ g_2(r,v) &= \frac{E_2(v)}{r} + \frac{1}{4}\left[\dot{h}_0\right]^2 + \dot{h}_0\partial_v^2 h_0 \end{aligned} \tag{A.11}$$

At next order

$$\begin{aligned} h_3(r,v) =& \frac{1}{4r^3}\left\{\int_{-\infty}^{v} E_2(x)\partial_x h_0 \, dx - r\ h_0 \left[\dot{h}_0\right]^2\right\} \\ f_4(r,v) =& \frac{h_0^2(v)\left[\dot{h}_0\right]^2}{8r} + \frac{D(v)h_0(v)}{8r^2} - \frac{\dot{h}_0}{128r^3}\left(-12D(v) + \left[\dot{h}_0\right]^3\right) \\ g_4(r,v) =& \frac{E4(v)}{r} + \frac{5}{4}h_0(v)^2\left[\dot{h}_0\right]^2 + h_0(v)^3\partial_v^2 h0 \\ &+ \frac{\dot{h}_0}{8r^2}\left[D(v) + 4E_2(v)h_0(v)\right] + \frac{1}{16r^3}\left(E_2(v)\left[\dot{h}_0\right]^2 + D(v)\partial_v^2 h_0\right) \\ h_4(r,v) =& 0 \end{aligned} \tag{A.12}$$

where $D(v) = \int_{-\infty}^{v} E_2(x)\partial_x h_0 \, dx$

Finally at the next order

$$
\begin{aligned}
h_5(r,v) =& \frac{D_1(v)}{2r^2} \\
&+ \frac{1}{24r^3}\left[6\int_{-\infty}^{v} D_2(x)\ dx + 5\left\{\int_{-\infty}^{v} dz \int_{-\infty}^{z} dy \int_{-\infty}^{y} D_4(x)\ dx\right\}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + 4\left\{\int_{-\infty}^{v} dy \int_{-\infty}^{y} D_3(x)\ dx\right\}\right] \\
&+ \frac{1}{r^4}\left[\frac{5}{24}\left\{\int_{-\infty}^{v} dy \int_{-\infty}^{y} D_4(x)\ dx\right\} + \frac{1}{6}\left\{\int_{-\infty}^{v} D_3(x)\ dx\right\}\right] \\
&+ \frac{1}{8r^5}\left[\int_{-\infty}^{v} D_4(x)\ dx\right]
\end{aligned}
\tag{A.13}
$$

where

$$
\begin{aligned}
D_1(x) &= -h_0(x)^3\left[\partial_x h_0\right]^2 \\
D_2(x) &= E_4(x)\partial_x h_0 + \frac{1}{4}D(x)h_0(x)\partial_x h_0 + E_2(x)h_0(x)^2\partial_x h_0 \\
D_3(x) &= \frac{1}{8}\left[5D(x)\left[\partial_x h_0\right]^2 + 15E_2(x)h_0(x)\left[\partial_x h_0\right]^2 + 15D(x)h_0(x)\partial_x^2 h_0\right] \\
D_4(x) &= \frac{1}{8}\left[18D(x)E_2(x) + 5E_2(x)\left[\partial_x h_0\right]^3 + 7D(x)h_0(x)\partial_x^2 h_0\right]
\end{aligned}
\tag{A.14}
$$

and (this follows from energy conservation)

$$
\begin{aligned}
\dot{E}_2 &= \frac{1}{2}\dot{h}_0\partial_v^3 h_0 \\
\dot{E}_4 &= \frac{3}{8}D(v)\dot{h}_0 + \frac{\dot{h}_0}{2}\left[3E_2(v)h_0(v) + \left[\dot{h}_0\right]^3 + 4h_0(v)\dot{h}_0\partial_v^2 h_0 + 2h_0^2\partial_v^3 h_0\right]
\end{aligned}
\tag{A.15}
$$

It follows in particular that the the 'initial' condition for normalizable evolution at $v = \delta t$ is given, to leading order, by

$$
h(r,\delta t) = \frac{1}{8r^3}\int_{-\infty}^{v}\left(\int_{-\infty}^{x} dy\left(\partial_y h_0 \partial_y^3 h_0\right)\partial_x h_0(x)dx\right)
\tag{A.16}
$$

This initial condition is of order $\frac{\epsilon^3}{(\delta t)^3 r^3}$ i.e. of order $\frac{\epsilon}{\tilde{r}^3}$ where $\tilde{r} = \frac{r}{E_2}$. This demonstrates that, for $v > \delta t$, our solution is a small perturbation about the black brane of energy density E_2.

A.5 Late times resummed perturbation theory

To leading order, the initial condition for the normalizable evolution of resummed perturbation theory for the field $h(r,v)$ is given by

$$
h(\delta t) = \frac{1}{4r^3}\left(\int_{-\infty}^{\delta t} E_2(x)\partial_x h_0\ dx\right) \equiv \frac{h_3^0(\delta t)}{r^3}
$$

Now, at the linearized level the equation of motion for the function h is simply the minimally coupled scalar equation. It follows that the subsequent evolution of the field h is simply given by

$$h = \frac{h_3^0(\delta t)}{M}\psi\left(\frac{r}{M^{\frac{1}{3}}}, (v - \delta t)M^{\frac{1}{3}}\right) \tag{A.17}$$

where the universal function ψ was defined in section 2. As in section 2, this perturbation is small initially, and at all subsequent times, justifying the resummed perturbation procedure.

B Generalization to arbitrary dimension

B.1 Translationally invariant scalar collapse in arbitrary dimension

In this subsection we will investigate how the results of section 2, which were worked out for the special case $d = 3$, generalize to $d \geq 3$. The mathematical problem we will investigate in this appendix was already set up in general d in subsection 2.1. It turns out that the dynamical details of collapse processes in odd and even dynamics are substantially different, so we will deal with those two cases separately.

B.1.1 Odd d

The general structure of the solutions that describe collapse in odd $d \geq 5$ is similar in many ways to the solution reported in section 2. The energy conservation equations may be studied via a large r Graham Fefferman expansion closely analogous to that described in section 2. The functions ϕ f and g may be expanded at large r as

$$\begin{aligned} \phi(r, v) &= \sum_{n=0}^{\infty} \frac{A_\phi^n(v)}{r^n} \\ f(r, v) &= r\left(\sum_{n=0}^{\infty} \frac{A_f^n(v)}{r^n}\right) \\ g(r, v) &= r^2\left(\sum_{n=0}^{\infty} \frac{A_g^n(v)}{r^n}\right) \end{aligned} \tag{B.1}$$

For $n \leq d-1$ the equations of motion locally determine $A_\phi^n(v)$, $A_f^n(v)$ and $A_g^n(v)$ in terms of $\phi_0(v)$. Each of these functions is a local expression (of n^{th} order in v derivatives) of $\phi_0(v)$. However local analysis does not determine $A_g^d(v) \equiv M(v)$ and $A_\phi^d(v) \equiv L(v)$ in terms of $\phi_0(v)$. $M(v)$ and $L(v)$ are however constrained to obey an energy conservation equation that takes the form

$$\dot{M} = k\dot{\phi}L(v) + \text{local} \tag{B.2}$$

where k is a constant and 'local' represents the a set of terms built out of products of derivatives of $\phi_0(v)$ that we will return to below. As in $d = 3$, $L(v) = \mathcal{O}(\epsilon^3)$, so the first term in (B.2) does not contribute at lowest order of the amplitude expansion of interest to this paper. The local terms in this equation (B.2) are easily worked out at lowest order, $\mathcal{O}(\epsilon^2)$, in the amplitude expansion, and we find $M(v) = C_2(v) + \mathcal{O}(\epsilon^4)$ with

$$C_2(v) = -\frac{2^{d-2}}{(d-2)}\left(\frac{(\frac{d-1}{2})!}{(d-1)!}\right)^2 \int_{-\infty}^{v} dt \left[\left(\partial_t^{\frac{d+3}{2}}\phi_0\right)\left(\partial_t^{\frac{d-1}{2}}\phi_0\right) - \frac{d-3}{d-1}\left(\partial_t^{\frac{d+1}{2}}\phi_0\right)^2\right] \quad \text{(B.3)}$$

$$C_2 = \frac{2^{d-1}}{(d-1)}\left(\frac{(\frac{d-1}{2})!}{(d-1)!}\right)^2 \int_{-\infty}^{\infty} dt \left(\partial_t^{\frac{d+1}{2}}\phi_0(t)\right)^2 \sim \frac{\epsilon^2}{(\delta t)^d}, \quad \text{(B.4)}$$

the generalization of (2.20) and (2.21) to arbitrary odd d. (B.4) gives the leading order expression for the mass of the black brane that is eventually formed at the end of the thermalization process.

Let us now turn to the naive amplitude expansion in arbitrary odd d. The first term in this expansion, ϕ_1 is easily determined and we find

$$\phi_1(r,v) = \sum_{k=0}^{\frac{d-1}{2}} \frac{2^k}{k!}\frac{(\frac{d-1}{2})!}{(d-1)!}\frac{(d-1-k)!}{(\frac{d-1-2k}{2})!}\frac{\partial_v^k\phi_0}{r^k} \quad \text{(B.5)}$$

Equations (2.12) then immediately determine f_2 and g_2. Turning to higher orders, it is possible to demonstrate that

1. The functions ϕ_{2n+1}, g_{2n} and f_{2n} have the following analytic structure in the variable r

$$\begin{aligned}
\phi_{2n+1}(r,v) &= \sum_{k=0}^{\frac{(2n+1)(d-1)}{2}-p(n)} \frac{\phi^k_{2n+1}(v)}{r^{\frac{(2n+1)(d-1)}{2}-k}} \\
f_{2n}(r,v) &= r\left(\sum_{k=0}^{n(d-1)-f(n)} \frac{f^k_{2n}(v)}{r^{n(d-1)-k}}\right) \\
g_{2n}(r,v) &= -\frac{C_{2n}(v)}{r^{d-2}} + r\left(\sum_{k=0}^{n(d-1)-g(n)} \frac{g^k_{2n}(v)}{r^{n(d-1)-k}}\right)
\end{aligned} \quad \text{(B.6)}$$

where

$$p(n) = d, \quad (2n+1 \geq d), \qquad p(n) = 2n+1 \quad (2n+1 \leq d), \quad \text{(B.7)}$$
$$f(n) = d, \quad (2n \geq d), \qquad f(n) = 2n \quad (2n \leq d), \quad \text{(B.8)}$$
$$g(n) = d-1, \quad (2n \geq d-1), \qquad g(n) = 2n-1 \quad (2n \leq d). \quad \text{(B.9)}$$

2. The functions $\phi^k_{2n+1}(v)$, $f^k_{2n}(v)$ and $g^k_{2n}(v)$ are each functionals of $\phi_0(v)$ that scale like λ^{-k} under the scaling $v \to \lambda v$.

3. For $v > \delta t$ $f_2 = f_4 = 0$, $g_2 = -\frac{C_2}{r^{d-2}}$ and $g_4 = \frac{-C_4}{r^{d-2}}$. Further, effectively, $p(n) = d$, $f(n) = 2d$ and $g(n) = 2d-1$ for $v > \delta t$ (all additional terms present in (B.6) vanish at these late times). Moreover the functions $\phi^k_{2n+1}(v)$, $f^k_{2n}(v)$ and $g^k_{2n}(v)$ are all polynomials in v whose degrees are bounded from above by $n+k-1$, $n+k-3$ and $n+k-4$ respectively.

As in $d = 3$, the polynomial growth in v of the coefficients of the naive perturbative expansion invalidates this expansion for large enough v. More specifically, the sums over k and n in the expressions above are weighted by rv and $\frac{\epsilon^2 v}{r^{d-1}}$ respectively. In the neighborhood of the horizon, $r \sim r_H \sim T \sim \frac{\epsilon^{\frac{2}{d}}}{\delta t}$ each of these sums is effectively weighted by the factor vT. Consequently, naive perturbation theory fails at times large compared to the inverse temperature of the brane. At times of order δt and for $r \sim r_H$ the sum over k and n are each weighted effectively by $\epsilon^{\frac{2}{d}}$. More generally, naive perturbation theory is good at times of order δt provided $r\delta t \gg \epsilon^{\frac{2}{d-1}}$, a condition that is satisfied at the event horizon.

As in $d = 3$ the IR divergence of the naive perturbation expansion has a simple explanation. Even within the validity of the naive perturbation expansion, the spacetime is not well approximated by empty AdS space, but rather by the Vaidya metric (1.1). The naive expansion, which may be carried out with comparative ease up to $v = \delta t$, may be used to supply initial conditions for the subsequent unforced normalizable evolution for resummed perturbation theory. For $v \geq \delta t$, the spacetime metric is given, to leading order, by the Vaidya form (1.1), with $C_2(v)$ given by the constant C_2 listed in (B.4)

Consequently, the spacetime metric for $v \geq \delta t$ is the black brane metric with temperature of order $\frac{\epsilon^{\frac{2}{d}}}{\delta t}$, perturbed by a propagating ϕ field and consequent spacetime ripples. The initial conditions at $v = \delta t$, that determine these perturbations at later times, are given to leading order in ϵ (read off from the most small r singular term in ϕ_3) as

$$\phi(r, v) = \frac{A}{r^{\frac{3(d-1)}{2}}}$$

where

$$A = \frac{(d-1)^2}{2(d-2)} \int_{-\infty}^{\infty} dt \Bigg[(d-2) \left(2^{\frac{d-1}{2}} \frac{\left(\frac{d-1}{2}\right)!}{(d-1)!} \right) C_2(t) \left(\partial_t^{\frac{d-1}{2}} \phi_0 \right) - \left(2^{\frac{d-1}{2}} \frac{\left(\frac{d-1}{2}\right)!}{(d-1)!} \right)^3 \left(\partial_t^{\frac{d-1}{2}} \phi_0 \right)^2 \left(\partial_t^{\frac{d+1}{2}} \phi_0 \right) \Bigg] \tag{B.10}$$

In terms of the normalized variable $x = \frac{r}{M^{\frac{1}{d}}}$ and $y = vM^{\frac{1}{d}}$ this initial condition takes the form

$$\phi(x) \sim \frac{\epsilon^{\frac{3}{d}}}{x^{\frac{3(d-1)}{2}}} \tag{B.11}$$

It follows that the solution at $v \geq \delta t$ is (in the appropriate x, y coordinates) an order $\epsilon^{\frac{3}{d}}$ perturbation about the uniform black brane. The coefficient of this perturbation is bounded for all y, and decays exponentially for large y over a time scale of order unity in that variable. The explicit form of the solution for ϕ, for $v > \delta t$, may be obtained in terms of a universal function, $\psi_d(x, y)$ as in section 2. The equation that we need to solve is

$$\partial_x \left(x^{d+1} \left(1 - \frac{1}{x^d} \right) \partial_x \psi_d \right) + 2x^{\frac{d-1}{2}} \partial_x \partial_y \left(x^{\frac{d-1}{2}} \psi_d \right) = 0 \tag{B.12}$$

As in section 2, this universal function appears to be difficult to obtain analytically, but is easily evaluated numerically. As an example in figure 5 we present a numerical plot

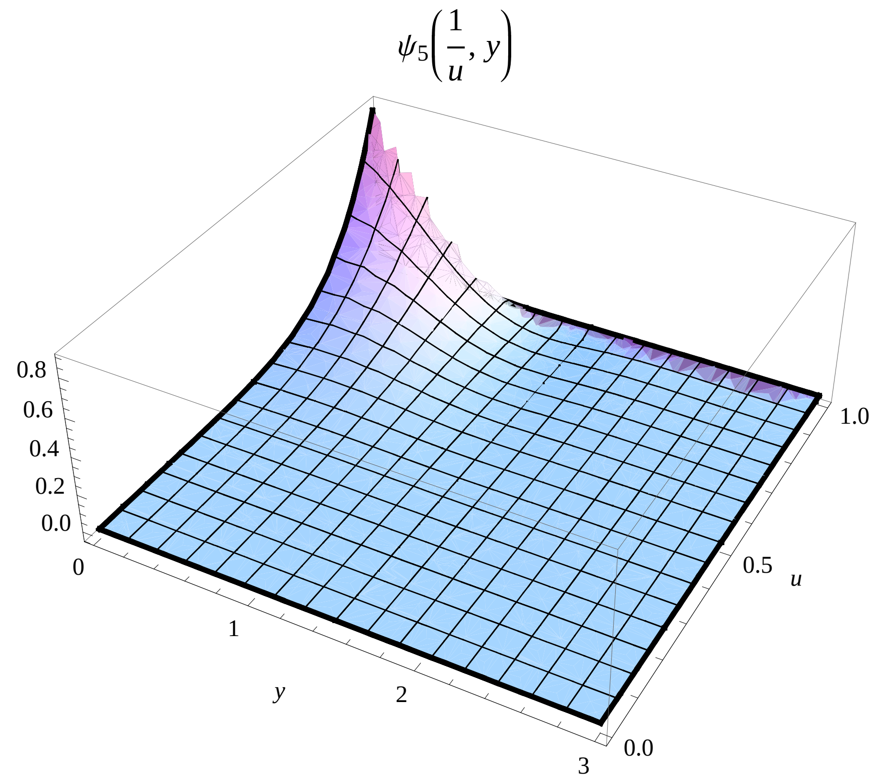

Figure 5. Numerical solution for the dilaton at late time in $d = 5$

of this function in $d = 5$. As in section 2 we find it convenient to display the numerical output for the function $\psi_5(\frac{1}{x}, y)$ over the full exterior of the event horizon, $u \in (0, 1)$.[25] In figure 6 we present a graph of $\psi_5(\frac{1}{0.7}, y)$ (i.e. as a function of time at a fixed radial location) Notice that this graph decays, roughly exponentially for $v > 0.5$ and that this exponential decay is dressed with a sinusodial osciallation, as expected for quasinormal type behavior. A very very rough estimate of this decay constant is provided by the equation ω_I using the equation $\frac{\psi_5(\frac{1}{0.7}, 1)}{\psi_5(\frac{1}{0.7}, .5)} = e^{-0.5\omega_I}$ which gives $\omega_I \approx 8.2T$ (here T is the temperature of our black brane given by $T = \frac{4\pi}{5}$). This number is the same ballpark as the decay constants for the first quasi normal mode of the uniform black brane reported in [40] (unfortunately those authors have not reported the precise numerical value for $d = 5$) .

B.1.2 Even d

In our analyses above we have so far focused attention on odd d (recall that d is the spacetime dimension of the dual field theory). In this subsection we will study how our results generalize to even d. While all the broad qualitative conclusions of the odd d analysis plausibly continue to apply, several intermediate details are quite different.The

[25]In order to obtain this plot, as in 2, we worked with the redefined field $\chi_5(u, y) = (1 - u)\psi_5(\frac{1}{u}, y)$ and imposed Dirichlet boundary conditions on this field at $u = 0$ and $u = 0.999999$. We also imposed the initial conditions $\chi_5 = (0.999999 - u)u^6$. The figure 5 was outputted by Mathematica-6's partial differential equation solver, with a step size of 0.0005 and an accuracy goal of 0.001.

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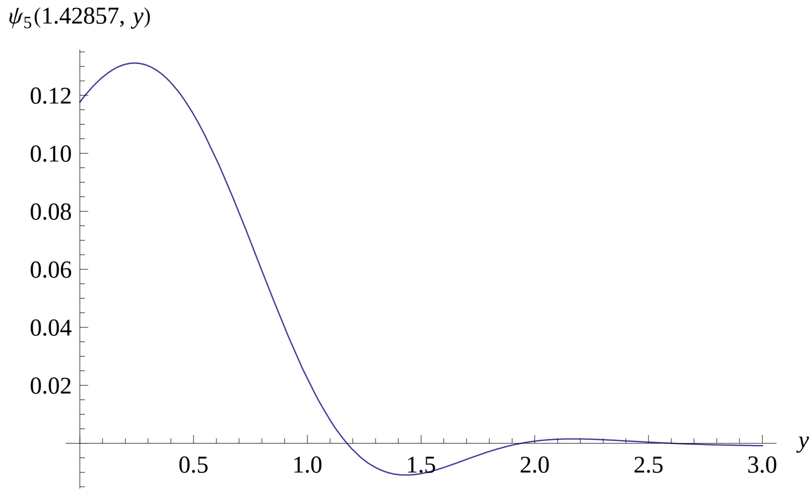

Figure 6. A plot of $\psi_5(\frac{1}{0.7}, y)$ as a function of y

analysis of all equations is more difficult in even than in odd dimensions. In this appendix we aim only to initiate a serious analysis of these equations, and to carry this analysis far enough to have a plausible guess for the behavior of our system. We leave a systematic analysis of these equations to future work.

The qualitative differences between even and odd d show themselves already in the Graham Fefferman expansion. We illustrate this by working out this expansion in $d = 4$. In this dimension the expansion of f, g, ϕ at large r take the form

$$\begin{aligned}
f(r,v) &= r - \frac{(\dot\phi_0)^2}{12r} - \frac{\ddot\phi_0\dot\phi_0}{36r^2} + \frac{-3(\dot\phi_0)^4 + 2\dddot\phi_0\dot\phi_0 - (\partial_v^2\phi_0)^2}{288r^3} \\
&\quad + \frac{-19\ddot\phi_0(\dot\phi_0)^3 - 1440L(v)\dot\phi_0 - 18\partial_v^4\phi_0\partial_v\phi_0 + 45\ddot\phi_0\dddot\phi_0}{21600r^4} \\
&\quad - \frac{\log(r)\dot\phi_0\left(\partial_v^4\phi_0 - 2(\dot\phi_0)^2\partial_v^2\phi_0\right)}{240r^4} + \dots \\
g(r,v) &= r^2 - \frac{5}{12}(\dot\phi_0)^2 - \frac{M(v)}{r^2} + \frac{\log(r)\left(-(\dot\phi_0)^4 + 2\dddot\phi_0\dot\phi_0 - (\partial_v^2\phi_0)^2\right)}{24r^2} + \dots \\
\phi(r,v) &= \phi_0 + \frac{\dot\phi_0}{r} + \frac{\partial_v^2\phi_0}{4r^2} + \frac{\frac{5}{36}(\dot\phi_0)^3 - \frac{1}{12}\dddot\phi_0}{r^3} + \frac{L}{r^4} + \dots \\
&\quad + \frac{\log(r)\left(\partial_v^4\phi_0 - 2(\dot\phi_0)^2\partial_v^2\phi_0\right)}{16r^4}
\end{aligned} \tag{B.13}$$

The energy conservation equation is

$$\dot M = \frac{1}{144}\left(40\ddot\phi_0(\dot\phi_0)^3 - 192L(v)\dot\phi_0 - 17\partial_v^4\phi_0\dot\phi_0 + 6\ddot\phi_0\dddot\phi_0\right) \tag{B.14}$$

and at quadratic order in ϵ we have

$$\begin{aligned}
M(v) &= C_2(v) + \mathcal{O}(\epsilon^4) \\
C_2(v) &= \frac{1}{144}\int_{-\infty}^{v} dt\left(-192L(t)\dot\phi_0 - 17\ddddot\phi_0\dot\phi_0 + 6\partial_t^2\phi_0\dddot\phi_0\right)
\end{aligned} \tag{B.15}$$

Unlike in even dimensions, it turns out that in odd dimensions $L(v)$ is nonzero at order ϵ. This is fortunate, as all the local terms in (B.15) are total derivatives, and so vanish when v is taken to be larger than δt. The full contribution to the mass of the black brane that is eventually formed from our collapse process arises from the term in (B.15) that is proportional to $L(v)$. As a consequence, the mass of the eventual black brane is not determined simply by Graham Fefferman analysis, but requires the details of the full dynamical process. These details may be worked out at lowest order in the ϵ expansion, (see below) and we will find

$$L(v) = \left(\frac{-7 + 12\log 2}{192}\right)\partial_v^4\phi_0 + \frac{1}{16}\int_{-\infty}^{v} dt \log(v-t)\ \partial_t^5\phi_0(t) + \mathcal{O}(\epsilon^3) \tag{B.16}$$

Plugging into (B.15) we find that $C_2(v)$ reduces to the constant C_2 for $v > \delta t$, and we have

$$C_2 = -\frac{1}{12}\int_{-\infty}^{\infty} dt_1 dt_2 \left(\partial_{t_1}^3\phi_0(t_1)\log(t_1-t_2)\Theta(t_1-t_2)\partial_{t_2}^3\phi_0(t_2)\right) \tag{B.17}$$

Let us now turn to the amplitude expansion of our solutions. We will work this expansion out only at leading order; already the leading order solution turns out to have qualitative differences (and to be much harder to determine and manipulate) than the corresponding solution in odd d.

Recall that ϕ_1 (B.5) is extremely simple when d was odd. To start with, the solution is local in time, i.e. $\phi_1(r, v_0)$ is completely determined by the value, and a finite number of derivatives, of $\phi_0(v_0)$. Relatedly $\phi(r, v)$ has a very simple analytic expression in r; it is a polynomial in $\frac{1}{r}$ of degree $\frac{d-1}{2}$. In even d, on the other hand the dependence of $\phi_1(r, v)$ on $\phi_0(v)$ is not local in time. Relatedly, the expansion of $\phi_1(r, v)$ in a power series in $\frac{1}{r}$ has terms of every order in $\frac{1}{r}$. Explicitly we find

$$\begin{aligned}
\phi_1(r,v) &= \int_0^\infty \partial_v^{d+1}\phi_0(v-t)\left(\frac{h(rt)}{r^d}\right)dt \\
h(x) &= \int_0^x dy \frac{(y(y+2))^{\frac{d-1}{2}}}{(d-1)!} \\
&= (-1)^{\frac{d}{2}}\binom{d}{\frac{d}{2}}\frac{\theta}{2^d} + \frac{1}{2^{d-1}}\sum_{k=0}^{\frac{d}{2}-1}\frac{(-1)^k}{d-2k}\binom{d}{k}\sinh\left((d-2k)\theta\right) \\
&\text{where} \quad \cosh\theta = 1 + x
\end{aligned} \tag{B.18}$$

Note that the function $h(x)$ admits the following large x expansion

$$\begin{aligned}
h(x) =& \frac{x^d}{(d-1)!} + \sum_{k=1}^{d-1}\frac{x^{d-k}}{(d-k)k!(d-1)!}\left(\prod_{m=1}^{k}(d-2m-1)\right) \\
&+ \frac{(-1)^{\frac{d}{2}+1}(d)!}{(d-1)!2^d((\frac{d}{2})!)^2}\left(\sum_{p=0}^{\frac{d}{2}-1}\frac{1}{(d-2p)(d-2p-1)}\right) + \frac{(-1)^{\frac{d}{2}}(d)}{2^d\left(\frac{d}{2}!\right)^2}\ln(2x) + \mathcal{O}\left(\frac{\ln x}{x}\right)
\end{aligned} \tag{B.19}$$

The fact that $h(x)$ grows (rather than decays) with x may cause the reader to worry that $\phi_(r,v)$ blows up at large v. That this is not the case may be seen by noting that $v^k\partial_v^{d+1}\phi_0$ may be rewritten as a sum of total derivatives when $k \leq d+1$ and so integrates to zero when $v > \delta t$ (in general it integrates to a simple local expression even for $v < \delta t$). Explicitly, plugging (B.19) into (B.18) and integrating by parts we find that $\phi_1(r,v)$ has the following large rt behavior

$$
\begin{aligned}
\phi_1(r,v) =& \sum_{i=0}^{d} \frac{A_i(v)}{r^i} + \frac{B(v)\ln(r)}{r^d} + \mathcal{O}(\frac{\ln r}{r^{d+1}}) \\
=& \,\phi_0(v) \\
&+ \sum_{k=1}^{d-1} \frac{\partial_v^k \phi_0(v)}{r^k} \left[\frac{(d-k-1)!}{k!(d-1)!} \left(\prod_{m=1}^{k} (d-2m-1) \right) \right] \\
&+ \frac{\partial_v^d \phi_0(v)}{r^d} \left[\frac{(-1)^{\frac{d}{2}+1}(d)!}{(d-1)!2^d((\frac{d}{2})!)^2} \left(\sum_{p=0}^{\frac{d}{2}-1} \frac{1}{(d-2p)(d-2p-1)} \right) \right] \\
&+ \int_0^\infty dt \frac{\partial_v^{d+1}\phi_0(v-t)}{r^d} \ln(2rt) \left[\frac{(-1)^{\frac{d}{2}}(d)}{2^d\left(\frac{d}{2}!\right)^2} \right] \\
&+ \mathcal{O}\left(\frac{\ln(r)}{r}^{d+1} \right)
\end{aligned}
\tag{B.20}
$$

(where the functions $A_i(v)$ and $B(v)$ are defined by this equation). On the other hand at small x we have

$$
h(x) = \frac{(2x)^{\frac{d+1}{2}}}{(d+1)(d-1)!}(1 + \mathcal{O}(x)) \tag{B.21}
$$

from which it follows that

$$
\phi_1(r,v) = \frac{1}{r^{\frac{d-1}{2}}} \frac{1}{(d+1)(d-1)!} \int_{-\infty}^{v} dt (2(v-t))^{\frac{d+1}{2}} \partial_t^{d+1}\phi_0(t) + \mathcal{O}\left(\frac{1}{r^{\frac{d-3}{2}}} \right), \tag{B.22}
$$

an expression that is valid at small rv. Note, in particular, that for $\delta t \ll v$, (B.22) reduces to

$$
\phi_1(r,v) = \frac{2^{\frac{d+1}{2}} \int_0^{\delta t} \phi_0(t)dt}{r^{\frac{d-1}{2}} v^{\frac{d+1}{2}}} \frac{1}{(d+1)(d-1)!} + \mathcal{O}\left(\frac{1}{r^{\frac{d-3}{2}}} \right) + \mathcal{O}\left(\frac{1}{t^{\frac{d+3}{2}}} \right) \tag{B.23}
$$

In particular this formula determines the behavior of the field ϕ_1 in the neighborhood of the event horizon $r_H \sim T$ for times that are large compared to δt but small compared to T^{-1}.

The functions f_2 and g_2 are easily expressed in terms of the function ϕ_0. We find

$$
\begin{aligned}
f_2(r,v) =& -\frac{1}{2(d-1)} \left[r \int_r^\infty (\partial_\rho \phi_1)^2 d\rho - \int_r^\infty \rho^2 (\partial_\rho \phi_1)^2 d\rho \right] \\
g_2(r,v) =& -\left(2\partial_v f_2(r,v) + (d-2) r f_2(r,v) + r^2 \partial_r f_2(r,v) \right) \\
&+ \frac{d(d-1)}{r^{d-2}} \int_0^r \rho^{d-2} f_2(\rho,v) d\rho - \frac{D_2(v)}{r^{d-2}}
\end{aligned}
\tag{B.24}
$$

The function $D_2(v)$ is determined by the requirement that the coefficient of $\frac{1}{r^{d-2}}$, in the large r expansion of $g_2(r,v)$ is $-C_2(v)$ (see (B.15)); in particular, for $v > \delta t$, $D_2(v) = C_2(v)$. At small r and for $v > \delta t$

$$\begin{aligned}
f_2(r,v) &= -\frac{K^2(v)}{2(d-1)(d-2)(d-3)r^{d-2}} + \mathcal{O}(\frac{1}{r^{d-3}}) \\
g_2(r,v) &= -\frac{C_2}{r^{d-2}} + \frac{\partial_v K^2(v)}{(d-1)(d-2)(d-3)r^{d-2}} + \mathcal{O}(\frac{1}{r^{d-3}}) \\
K(v) &= \frac{1}{(d+1)(d-1)!}\int_{-\infty}^{v} dt (2(v-t))^{\frac{d+1}{2}} \partial_t^{d+1}\phi_0(t) \\
&\approx \frac{2^{\frac{d+1}{2}}\int_0^{\delta t}\phi_0(t)dt}{v^{\frac{d+1}{2}}} \frac{1}{(d+1)(d-1)!} \quad (v \gg \delta t)
\end{aligned} \tag{B.25}$$

We would like to draw attention to several aspects of these results. First note that $\phi_1(r,v)$ is small provided $(r\delta t)^{\frac{d-1}{2}} \gg \epsilon$. Consequently, we expect a perturbative analysis to correctly capture the dynamics of our situation over this range of coordinates; note that this is exactly the same estimate as for odd d. Next note that the maximal singularity, at small r, in the functions f_2 and g_2, are both of order $\frac{1}{r^{d-2}}$; this is the same as the maximal singularity in the analogous functions in odd d (see the previous subsection). As the function $g_0(r,v) = r^2$, it follows, as in the previous function, that our spacetime metric is not uniformly well approximated by the empty AdS space over the full range of validity of perturbation theory. Over this entire range, however, it is well approximated by a Vaidya type metric, where the mass function for this metric is given at leading order by the coefficient of $-\frac{1}{r^{d-2}}$ in $g_2(r,v)$ above.

Unlike the situation in odd dimensions, the leading order mass function $M(v)$, in the effective Vaidya metric, is not given simply by $C_2(v)$. In particular, when $v \gg \delta t$ we have from (B.25) that

$$\frac{C_2 - M(v)}{C_2} \sim \left(\frac{\delta t}{v}\right)^{d+2}.$$

In other words, the leading order metric for the thermalization process, in even d, is not given precisely by the metric of the uniform black brane for $v > \delta t$. However it decays, in a power law fashion, to the black brane metric at times larger than δt. As a consequence at times $\delta t \ll v \ll T^{-1}$ the leading order metric that captures the thermalization process is arbitrarily well approximated by the metric of a uniform black brane. It follows that, while the spacetime described in this subsection does not capture the dual of instantaneous field theory thermalization (as was the case in odd d), it yields the dual of a thermalization process that occurs over the time scale of the forcing function rather than the much longer linear response time scale of the inverse temperature.

We will not, in this paper, continue the perturbative expansion to higher orders in ϵ. We suspect, however, that the computation of ϕ_3 when carried through will yield a term proportional to $\frac{\epsilon^3}{r^{\frac{3(d-1)}{2}}}$ that is constant in time. This term will dominate the decaying tail of $\phi_1(r,v)$ at a time intermediate between δt and T^{-1} and will set the initial condition for the late time decay of the ϕ field (over time scale T^{-1}) as was the case in odd dimensions. It would be very interesting to verify or correct this guess.

B.2 Spherically symmetric flat space collapse in arbitrary dimension

B.2.1 Odd d

The discussion of section 3 also extends to the study of spherically symmetric collapse in a space that is asymptotically flat $R^{d,1}$ for arbitrary odd d. In this section we will very briefly explain how this works, focussing on the limit $y = \frac{r_H}{\delta t} \gg 1$.

To lowest order in the amplitude expansion we find

$$\phi_1(r,v) = \sum_{m}^{\frac{d-3}{2}} 2^{\frac{d-3}{2}-m} \frac{(-1)^m}{m!} \frac{\left(\frac{d-3}{2}+m\right)!}{\left(\frac{d-3}{2}-m\right)!} \frac{\partial_v^{\frac{d-3}{2}-m}\psi(v)}{r^{\frac{d-1}{2}+m}} \tag{B.26}$$

Here $\psi(v)$ is a function of time that we take, as usual, to vanish outside $v \in (0, \delta t)$, and be of order $\epsilon_f(\delta t)^{\frac{d-1}{2}}$, where ϵ_f is a dimensionless number such that $\epsilon_f \gg 1$. As in section 3 the parameter that will justify the amplitude expansion will be $\frac{1}{\epsilon_f}$.

(B.26) together with constraint equations immediately yields an expression for the functions f_2 and g_2. In particular, the leading large r approximation to g_2 is given by

$$\begin{aligned} g_2(r,v) &= -\frac{M(v)}{r^{d-2}} \\ M(v) &= -\frac{2^{(d-4)}}{d-1}\int_{-\infty}^{v} dt \left[\left(\partial_t^{\frac{(d-3)}{2}}\psi(t)\right)\left(\partial_t^{\frac{(d+1)}{2}}\psi(t)\right) - \frac{d-3}{d-2}\left(\partial_t^{\frac{(d-1)}{2}}\psi(t)\right)^2 \right] \end{aligned} \tag{B.27}$$

Note that $\phi_1 \ll 1$ whenever $r^{\frac{d-1}{2}} \ll (\delta t)^{\frac{d-1}{2}}\epsilon_f$ so we expect the amplitude expansion to reliably describe dynamics over this range of parameters. As in section 3, however, g_2 cannot be ignored in comparison to $g_0 = 1$ throughout this parameter regime. As in section 3, this implies that our spacetime is well approximated by a Vaidya type metric rather than empty flat space even at arbitrarily small $\frac{1}{\epsilon_f}$. The mass function of this Vaidya metric is given by $M(v)$ in (B.27).

As in section 3 one may ignore this complication at early times $v \ll r_H$ over which the solution is well approximated by a naive perturbation expansion that uses empty flat space as its starting point. It is possible to demonstrate that this naive expansion has the following analytic structure in the variables r and v

- 1. The functions Φ_{2n+1}, F_{2n} and G_{2n} have the following analytic structure in the variable r

$$\begin{aligned} \Phi_{2n+1}(r,v) &= \sum_{m=0}^{\infty} \frac{\Phi_{2n+1}^m(v)}{r^{(2n+1)\frac{d-1}{2}+m}} \\ F_{2n}(r,v) &= r\sum_{m=0}^{\infty} \frac{F_{2n}^m(v)}{r^{n(d-1)+m}} \\ G_{2n}(r,v) &= -\delta_{n,1}\frac{M(v)}{r^{d-2}} + r\sum_{m=0}^{\infty} \frac{G_{2n}^m(v)}{r^{n(d-1)+m}} \end{aligned} \tag{B.28}$$

- 2. The functions $\Phi_{2n+1}^m(v)$, $F_{2n}^m(v)$ and $G_{2n}^m(v)$ are each functionals of $\psi(v)$ that scale like $\lambda^{m-(2n+1)\frac{d-3}{2}}$ $\lambda^{m-n(d-3)}$ and $\lambda^{m-n(d-3)-1}$ under the the scaling $v \to \lambda v$. $M(v)$ scales like λ^{2-d} under the same scaling.

- 3. For $v > \delta t$ the $\Phi^m_{2n+1}(v)$ is polynomials in v of degree $\leq n+m-1$; $F^m_{2n}(v)$ and G^m_{2n} are polynomials in v of degree $\leq n+m-3$ and $n+m-4$ respectively.

It follows that, say, $\phi(r,v)$, is given by a double sum

$$\phi(r,v) = \sum_n \Phi_{2n+1}(r,v) = \sum_{n,m=0}^{\infty} \frac{\Phi^m_{2n+1}(v)}{r^{(2n+1)\frac{d-1}{2}+m}}.$$

Now sums over m and n are controlled by the effective expansion parameters $\sim \frac{v}{r}$ (for m) and $\frac{\psi^2 v}{(\delta t)^{d-2} r^{d-1}} \sim \frac{v}{\delta t \epsilon_f^{\frac{2}{d-2}}} \sim \frac{v}{r_H}$ (for n; recall that in the neighborhood of the horizon $r_H^{d-2} \sim (\delta t)^{d-2}\epsilon_f^2$).

As in section 3, it follows that the naive perturbation expansion breaks down for times $v \gg r_H$. However this expansion is valid everwhere outside the event horizon at times of order δt, and so may be used to set the initial conditions for a resummed perturbation expansion that uses the Vaidya metric as its starting point. For $v > \delta t$ the mass function of the Vaidya metric reduces to a constant. At long times our solution is given by a small perturbation around a black hole of mass M. This perturbation is best analyzed in the coordinates $x = \frac{r}{M^{\frac{1}{d-2}}}$ and $y = \frac{v}{M^{\frac{1}{d-2}}}$. In these coordinates the leading order tail of ϕ, at long times, is given by motion about a black hole of unit Schwarzschild radius perturbed by the ϕ field with initial condition

$$\phi(x,0) = \frac{\phi^0_3(\delta t)}{M^{\frac{3(d-1)}{2(d-2)}} x^{\frac{3(d-1)}{2}}} \sim \frac{1}{\epsilon_f^{\frac{3}{d-2}}}$$

The smallness of this perturbation justifies linearized treatment of the subsequent dynamics.

B.2.2 Even d

We will not, in this paper, attempt an analysis of the spherically symmetric collapse to form a black hole asymptotically $R^{d,1}$ for even d. Here we simply note that the leading order large ϵ_f solution for $\phi_1(v)$ may formally be expressed as

$$\phi_1(r,v) = \int d\omega \left(q(\omega) e^{i\omega(v-r)} \frac{H^{(1)}_{\frac{d-2}{2}}(r\omega)}{r^{\frac{d-2}{2}}} \right) \qquad \text{(B.29)}$$

for any function $q(\omega)$ where $H_n(x)$ is the n^{th} Hankel function of the first kind, i.e.

$$H^{(1)}_n(x) \approx \sqrt{\frac{2}{\pi x}} \left(e^{i(x-\frac{\pi}{4}-\frac{n\pi}{2})} + \mathcal{O}\left(\frac{1}{x}\right)\right)$$

Using this expansion, it is easily verified that $\phi_1(r,v)$ reduces, at large r, to an incoming wave that takes the form $\frac{\psi(v)}{r^{\frac{d-1}{2}}}$. The evolution of this wave to small r is implicitly given by (B.29). It should be possible to mimic the analysis of subsubsection B.1.2 to explicitly express $\phi_1(r,v)$ as a spacetime dependent Kernel function convoluted against $\psi(v)$. In analogy with subsection B.1.2 it should also be possible to expand $g_2(r,v)$ about small r. It is tempting to guess that such an analysis would reveal that the leading singularity in $g_2(r,v)$ scales like $\frac{1}{r^{d-2}}$, so that the metric is well approximated by a spacetime of the Vaidya form. We leave the verification of these guesses to future work.

B.3 Spherically symmetric asymptotically AdS collapse in arbitrary dimension

It should be straightforward to generalize the analysis of section 4 to arbitrary odd d, and perhaps also to arbitrary even d. We do not explicitly carry out this generalization in this paper. However it is a simple matter to infer the various scales that will appear in this generalization using the intuition and results of subsections B.1 and B.2, and the fact that the results of global spherically symmetric AdS collapse must reduce to Poincare patch collapse in one limit and flat space collapse in another. We have reported these scales in the introductionto section 4.

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